REINFORCED CONCRETE MEMBERS IN BENDING, SHEAR AND TORSION

A THESIS

Submitted to the faculty of the Department of Civil Engineering. Indian Institute of Technology, Kanpur

by

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ABSTRACT

In this dissertation, a survey of what is known about,

(i) Prismatic plain concrete members subjected to pure torsion and

(ii) Reinforced concrete members with or without web reinforcement in pure torsion, torsion combined with bending or combined bending, torsion and shear,

is first presented. The behavioural aspects of members tested by various investigators are compared with elastic, plastic and Ultimate Equilibrium theories to find the range of their validity.

About 100 test-results conducted on the ultimate strength of R.C. members subjected to pure torsion, combined bending and torsion and torsion combined with bending and shear, by various researchers, are compared with Lessig's Ultimate Equilibrium Theory. Based on this analysis a direct ultimate design method has been suggested instead of the usual trial and error analysis of Lessig.

It is concluded that pure torsion and predominant torsional moment loading can cause sudden failures, and since the Ultimate Equilibrium Method, overestimates the capacity of such members, a capacity reduction factor of 0.8 is recommended. Lessig's analysis fails when there is no web reinforcement. For members without web reinforcement and subjected to combined loading, Hsu's simplified interaction surface and design criteria is promising.

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SECTION 1 : INTRODUCTION

1.1 INTRODUCTORY REMARKS :

In the reinforced concrete construction the phenomenon of torsion occurs quite frequently. In some cases, torsion has so insignificant an influence that it is not considered; in others, however, it requires special consideration in design. Torsion rarely occurs alone in the reinforced concrete structures; it is present more often in combination with the transverse shear and bending.

Torsion results mainly from the monolithic character of in-situ concrete construction. Thus an extra floor beam produces torsion in the spandrels which can be a primary effect if the wall offers little resistance to the torsional deformation of the spandrel (Fig. 1.1) An eccentrically placed wall produces torsion in the spandrel beams.(Fig.1.2) A secondary beam framing into the primary beam off the column produces torsion in the primary beam (Fig. 1.3). Torsion occurs in space frames and interconnected girders, and ingrid floor slabs. Consider two beams EB and DG and the column FC intersecting at A (Fig. 1.4) A load placed on AB produces bending in that member and torsion in the beam at right angles. Torsion in the column can result from an eccentric horizontal load acting on a building, as for example in a building partly sheltered from wind. (Fig. 1.5) The design of spiral staircase and Free standing stair-case (Fig. 1.10) introduces problem of torsion. Twisting moments may exercise a controlling influence over the design of major edge beams such as in two-way slabs on beams. The most practical example illustrating torsion as a primary effect is that of girders with horizontal projections. The frame shown (Fig. 1.6) would, if used in the horizontal position (Fig.1.7) be subjected to combined bending and torsion. Another example comes across when spandrel beams meet at the corner without a column, (Fig. 1.8) and in case of curved beams, (Fig. 1.9).

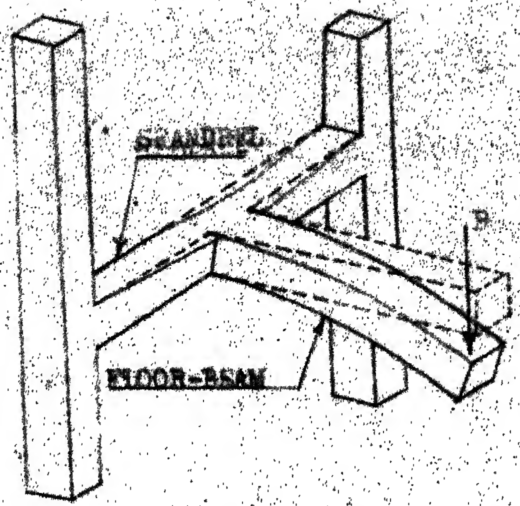


FIG. 1.1 A Floor Beam framing into the Spandrel between two Columns produces Torsion in the Spandrel
(From Ref. 46).

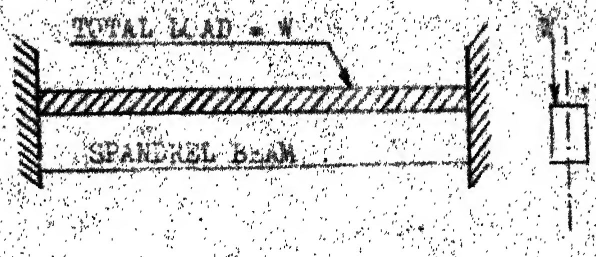


FIG. 1.2 An eccentrically placed wall produces Torsion in the Spandrel
(From Ref. 46).

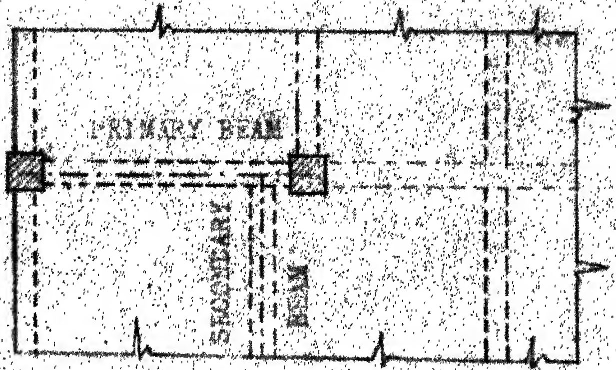


FIG. 1.3 A Secondary Beam off a Primary Beam produces torsion in Primary Beam (From Ref. 46).

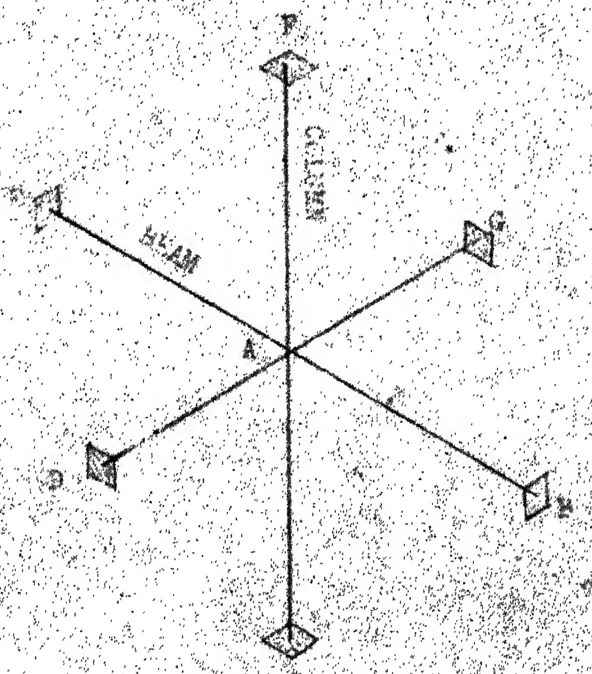


FIG. 1.4 Space Frame (From Ref. 46)

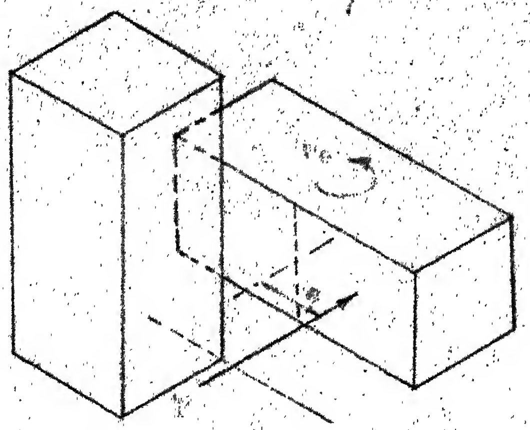


FIG. 1.5 Torsion in a Column due to an eccentric
horizontal load
(From Ref. 46)

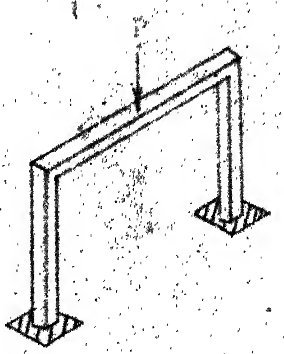


FIG. 1.6 Frame in a vertical
position
(From Ref. 46)

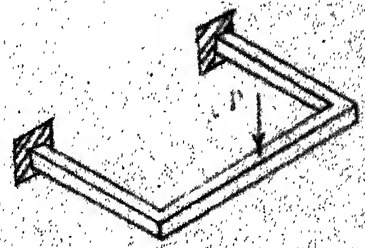


FIG. 1.7 Frame in horizontal
position
(From Ref. 46)

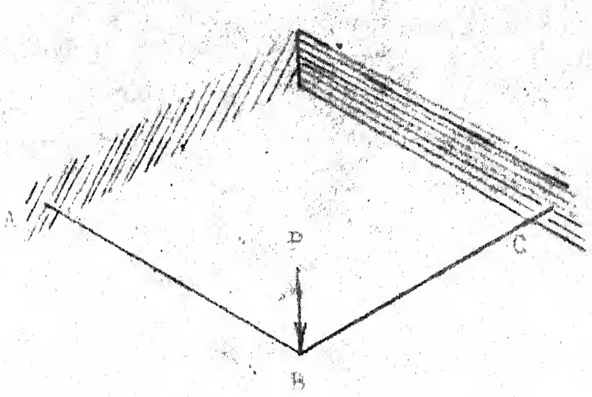


FIG. 1.8 Spandrel Beams meeting at a corner
without a Column support
(From Ref. 46)

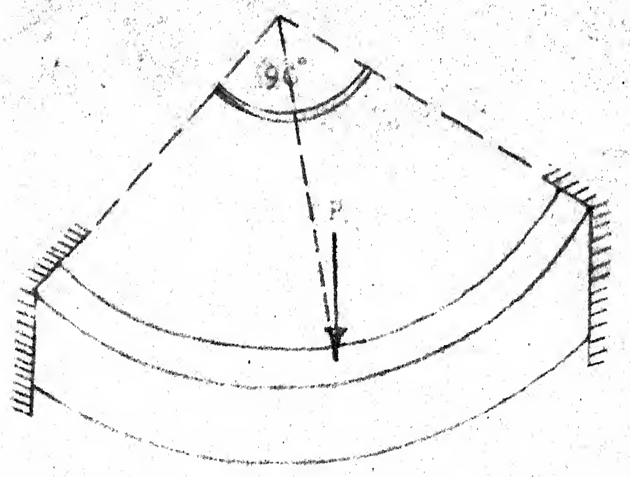


FIG. 1.9 Curved Beam

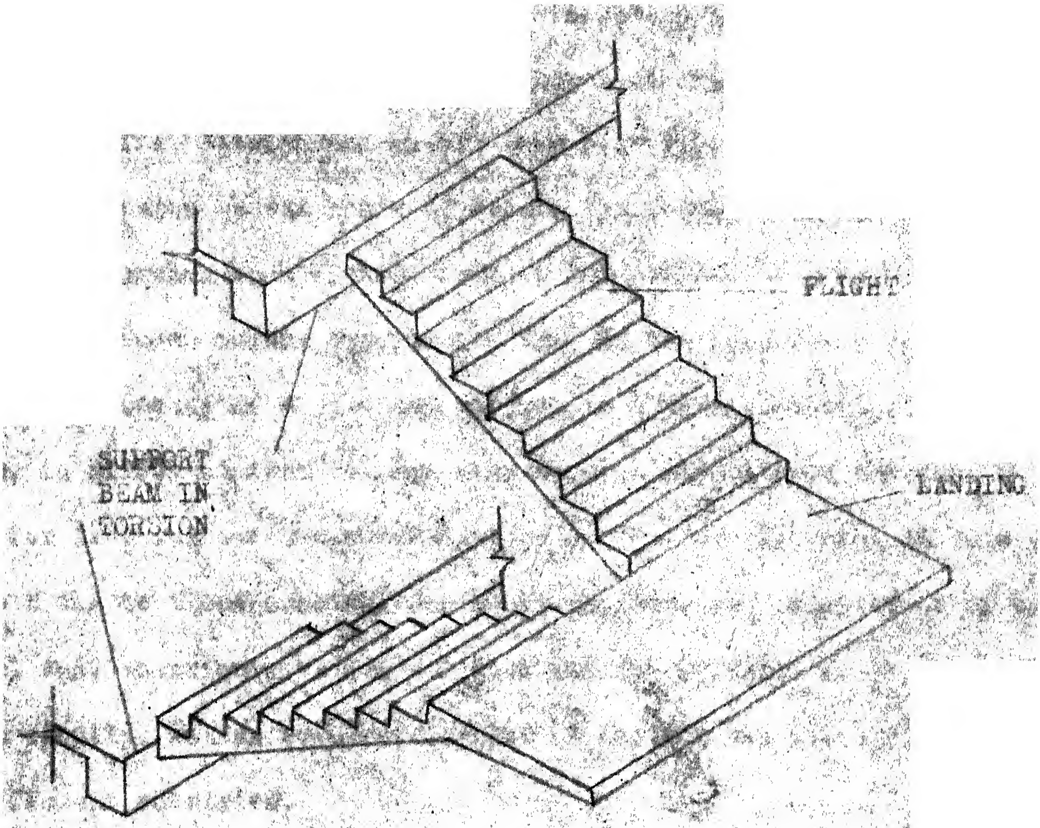


FIG. 1.10 Free-standing Stair without Landing Support

So we encounter the necessity of designing reinforced concrete members subject to combined bending, torsion and shear.

The revised Indian code of practice[64] does not contain a rational design method for specimens subjected to combined bending, torsion and shear. Several codes of practice, including that of India, Australia and America, are silent about the problem that arises out of the interaction between bending, torsion and shear; they just take Saint Venant's elastic theory of torsion as the basis of design for torsion and recommend the design for torsion as a separate effect independent of bending and shear; but this is found untrue experimentally. It has been found by tests that torsional capacity of a specimen subject to combined bending, torsion and shear, is different from the one when torsion acts alone; the same is true for flexural and shear capacities. So that we just can not take the effects due to these combined forces as separate and independent of each other. And, therefore, we can not just add the effects due to these forces algebraically, as has been recommended in most of the codes of practices of different countries.

In the case of pure flexure, we have already accepted the "Ultimate Load Method" as a more rational design-method, because it is based on the actual behaviour of the specimens loaded and tested upto failure. By this approach, we are sure of the strength of the member and we have to guard against crack-formation at working loads. In the same way, the 'Ultimate Equilibrium Method' is based on the actual behaviour of members subject to combined bending, torsion and shear. As in ultimate load method for pure flexure, we design members based on strength. Then such members have to be provided against excessive crack-formation at working loads. Both these aspects of design - strength and crack control -

are elaborately covered by the 'Ultimate Equilibrium Method' and this method has been incorporated in the new Russian code[67].

1.2 OBJECT AND SCOPE :

The object of this thesis is to develop a design method for a reinforced concrete beams subjected to combined bending, shear and torsion, based on ultimate load considerations. Towards the above-said end, the behaviour of plain and reinforced concrete members subject to pure torsion, must be first understood. So Sections 2 through 4 of this thesis deal with a review of the latest information available on plain and reinforced members subject to pure torsion. Section 5 is concerned with R.C. members subjected to combined bending and torsion. Section 6, which is the final objective of this work, is concerned with developing a design method for reinforced concrete members subject to combined bending, shear and torsion based on the ultimate equilibrium approach. Suggested design-procedure is given and a design example has been fully worked out as an illustration for the design steps suggested. The correctness of the ultimate equilibrium analysis has been shown to be good when the theoretical and experimental results are compared (Table Nos. 1 - 9; pp). In the middle of the progress of the research, a book on Reinforced and Prestressed Concrete in Torsion, written by Cowan[46] came to the attention of the author. This book covers the work upto 1964 and gives an exhaustive information on behaviour and strength of members, Codes of Practice in various countries and an extensive bibliography. Also a chapter on Ultimate Equilibrium Method written by I.M. Lyalin covering significant Russian research on the above topic. This thesis, therefore, emphasizes the research done after 1964 by research workers such as T.C. Hsu, Buchanan, Gesund, Sarkar and Evans, Goode and Helmy,

Vijayarangan and K.T.S. Iyenger, G.S. Pandit, Warawuk and Narayanswamy etc. Their experimental results are compared with the Ultimate Equilibrium Method and significant advance in the knowledge of behaviour under combined bending, torsion and shear is critically reviewed.

It has come to the notice of the author that the Symposium on Torsion was held by American Concrete Institute in 1966, the proceedings of which were not available at the time of writing the dissertation. However abstracts of the papers in the Symposium are given in Reference 47. The following are the papers read at the Symposium :

(1) Review of Activities of Committee 438, Torsion . . . . G.P. Fisher

(2) Aspects of Torsion in Concrete Design .... K.G. Tamberg and R.A. Shoolbred.

(3) A Look at the Development of Torsion Theories of Structural Concrete . . . . P.Z. Zia.

(4) Behaviour of Concrete Members Subject to Torsion . . . . T.T.C. Hsu and E.L. Kemp.

(5) How to Design for Torsion . . . . A.H. Mattock

Hsu's work has been extensively referred in the dissertation (References : 55 - 58). Kemp's previous work is also referred (Ref.37) in the thesis.

1.3 ACKNOWLEDGEMENTS :

The author is greatly indebted to his guide, Dr. J.K. Sridhar Rao, Assistant Professor in Civil Engineering Department, I.I.T., Kanpur with whose constant guidance, it has been possible to complete this thesis.

Dr. A.H. Shah, Assistant Professor in Civil Engineering Department, I.I.T., Kanpur , suggested the topic of this work as an area in which research needs to be undertaken and the author is thankful to him for his suggestion.

The author wishes to acknowledge with appreciation Shri G. L. Misra, who typed the manuscript; and author's wife, Sushma, who proof-read the manuscript.

1.4 NOTATION

a_1 = distance from the horizontal face of the member to the axis of the longitudinal bars near this edge.

a_2 = distance from the vertical face of the member to the axis of the longitudinal bars near this face

AB = neutral axis for the design of the three dimensional section in ultimate equilibrium method

b = width of the rectangular section

b' = width of the enclosed concrete section within the rectangular stirrups

C = torsional rigidity

C_1, C_2 = projections of the Neutral Axis

F = plastic stress function

F_{s1} = total cross-sectional area of all longitudinal reinforcement near the flextural tension face of the section of width 'b'

F_{s2} = the same near each of the faces of depth 'h'

f'_c = compressive cylinder strength of concrete

f_r = modulous of rupture for plain concrete

$f_{st.1}$ = cross-sectional area of one transverse bar near the side of width 'b'

$f_{st.2}$ = the same near the faces of depth 'h'

$f_{st.p}$ = cross-sectional area of the spiral wire

f_t = tensile strength of concrete

G = modulous of rigidity of concrete

h = depth of the rectangular section

h_o = effective depth of the section

h' = depth of the concrete section enclosed within the rectangular stirrups

K = ratio of applied torque to applied bending moment

L = length of the beam

M = applied bending moment

Q = applied shear force at the section

R_b = design strength of concrete in compression due to bending

T = applied torque

T_{pl} = plastic torsional resistance of the section

T_{up} = ultimate torsional resistance of plane concrete section

u_1, u_2 = spacing of the transverse bars (respectively near the faces of width 'b' and depth 'h')

u_p = pitch of the spiral reinforcement

V = volume under the sand-heap

W = a constant in finite difference method

x_1 = depth of neutral axis according to first failure scheme

x_2 = depth of the neutral axis according to second failure scheme

x, y, z = rectangular coordinates of a point

γ = shearing strain

θ = rotation or twist angle per unit length

θ_1 = total rotation angle in a length L units

θ_{up} = ultimate angle of rotation for plain concrete

σ_{s1} = design strength of longitudinal tension reinforcement

σ_{s2} = design resistance of longitudinal compression reinforcement

σ_{st} = design resistance of transverse reinforcement

$\sigma_{st.p}$ = yield stress of spiral reinforcement

τ = torsional shear stress

τ_{xz} = x-component of the torsional shear stress

τ_{yz} = y-component of the torsional shear stress

ν = a constant in elastic theory of torsion

SECTION 2 : PRISMATIC PLAIN CONCRETE SPECIMENS SUBJECTED TO PURE TORSION

2.1 BEHAVIOUR :

2.1.1 CONVEX-CROSS-SECTIONS (RECTANGULAR, CIRCULAR) : -

The experimental investigation of the behaviour of plain concrete reported by various investigators(37,57,58) show, without exception, that a specimen with a convex cross-section fails as soon as the first crack is formed. The failure is sudden, destructive and without warning. The behaviour upto failure is characterised by small detrusions and little evidence of distress in the concrete until the first crack develops and the specimen fails. The torque-twist curves for plain specimens, where reported in the literature show a nearly linear relationship between torque and the angle of twist upto approximately 80% of the torsional capacity of the specimen. Before failure sufficient in-elastic stress redistribution occurs to cause the torque-twist curves to curve slightly just prior to failure.

2.1.2 CONCAVEX-SECTIONS(T, L, I SECTIONS) :-

Zia(60) has shown that the first cracks in concave sections occurs in the web but the specimen continues to carry torque. The failure does not take place until the cracks propagate through the flanges; the cracks occurring at 45° to the longitudinal axis of the beam.

2.1.3 PLASTIC MODELS (RECT., CIRCULAR, AND T-SECTION)(54) :-

Buchanan (54) tested experimentally, three circular plastic model beams to obtain torque-twist curve and the results of his experiments are discussed in this sub-section. The tests were conducted on models made from Ultracal-30 Plaster. The mix consisted of the following proportions :-

Ultracal - 30	40% by weight
Petersburg sand (passing a # 30 sieve)	20% of dry weight
Crushed limestone (passing a $\frac{1}{4}$" sieve)	40% " " "
Water	35% of ultracal-30 by weight

The proportions of sand and limestone were chosen to provide compressive stress-strain properties of the hardened plaster that were nearly identical to the compressive stress-strain properties expected for the concrete prototype. Theoretical torque-twist ($T - \theta_1$) curves were calculated by the method of finite differences for these three beams and were found to be in good agreement with the experimental curves. One of the three ($T - \theta_1$) curves as obtained by Buchanan is shown (Fig. 2.1)

A rectangular beam was tested to obtain ($T - \theta_1$) data. A theoretical ($T - \theta_1$) curve based on an assumed plastic distribution of stress gave good agreement with the experimental data (Figs: 2.2, 2.3).

Also using strain-hardening property of the plastic used, the theoretical ($T - \theta_1$) curves for the strain-hardened stress distribution gave good agreement with the experimental data (Figs: 2.2, 2.4).

The experimental ($T - \theta_1$) curves for circular, rectangular and T-section plastic model beams as obtained by Buchanan are shown. All of these curves show that ($T - \theta_1$) curve is a non-linear curve (Figs: 2.1, 2.2, 2.5).

2.2 STRENGTH :-

In developing the analytical method for predicting the failure of a circular or a non-circular prismaticbeam, the following approaches have been used :

(i) Elastic Theory due to Saint Venant with Bach's approximation

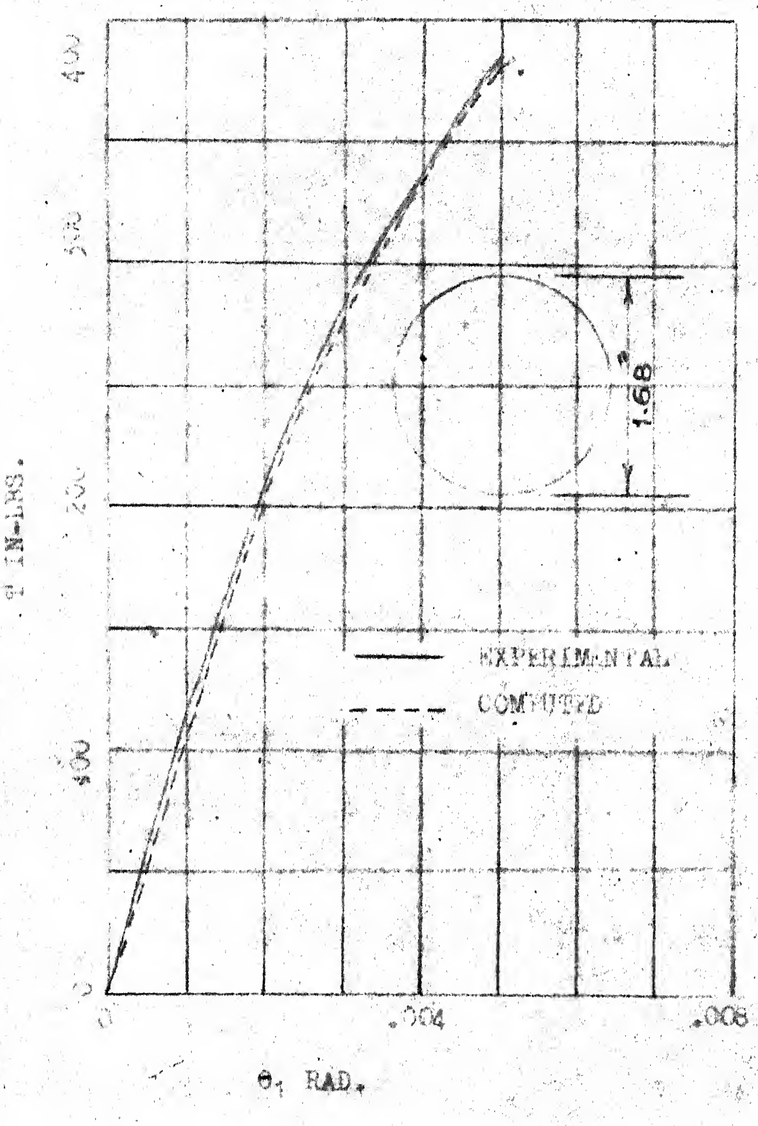


FIG. 2.1 Torque versus Rotation-Circular Beam
(From Ref. 54).

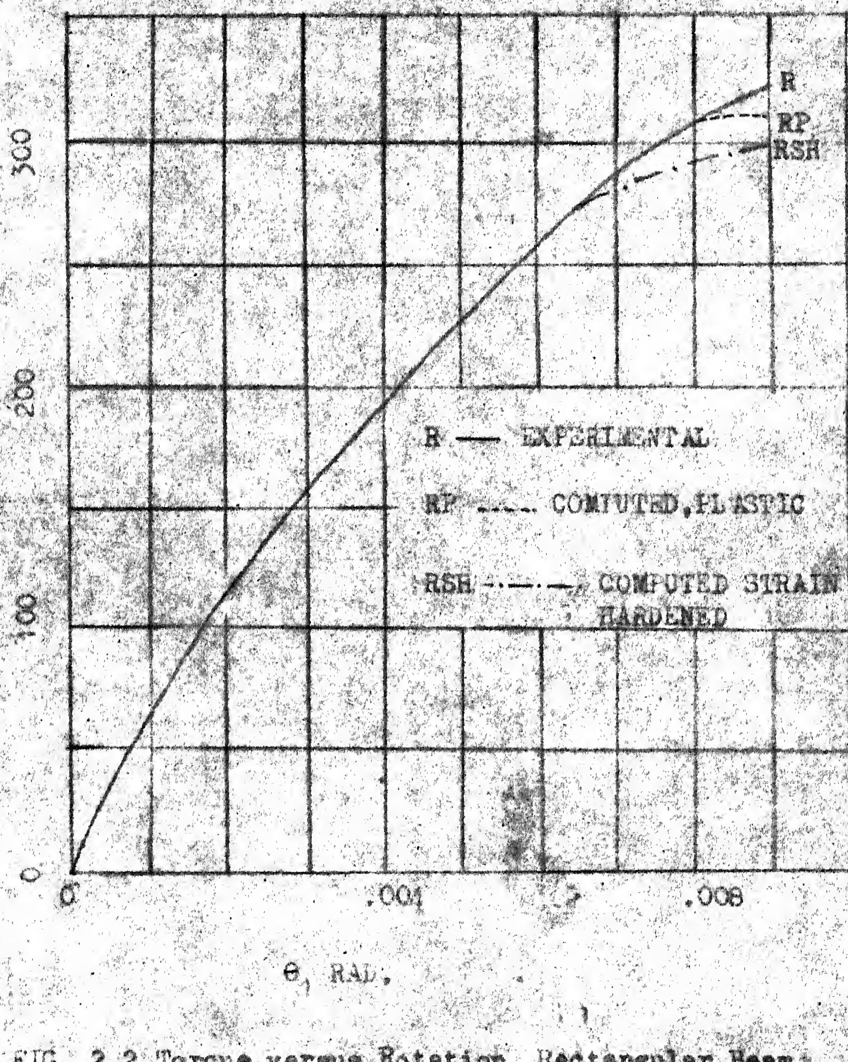


FIG. 2.2 Torque versus Rotation, Rectangular Beam (From Ref. 54).

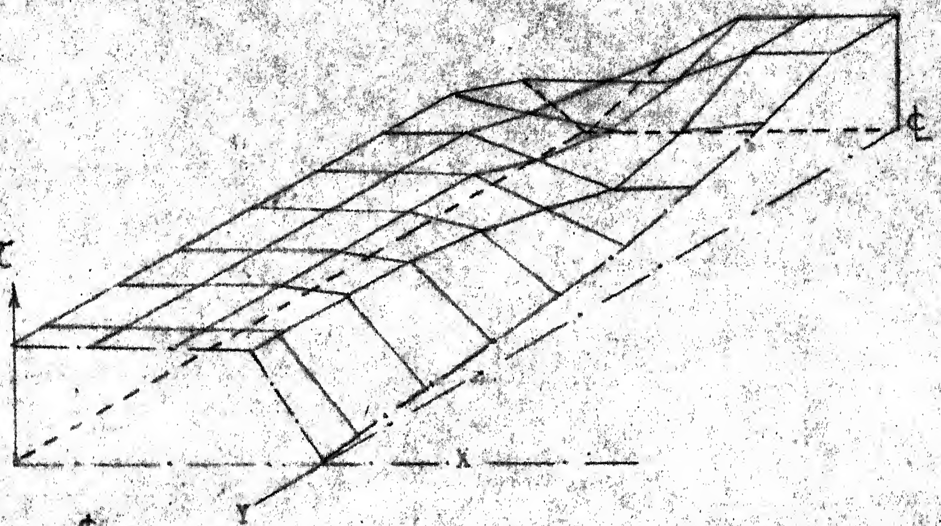


FIG. 2.3 Plastic

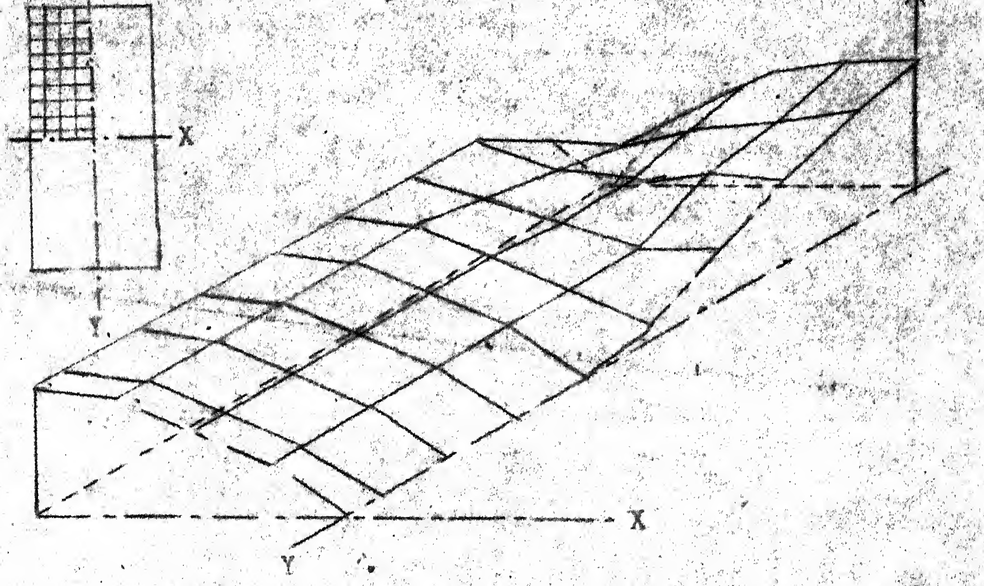


FIG. 2.4 Strain-hardened

Plots of Shearing Stress Distribution for One Quadrant of the Rectangular cross-section.
(From Ref. 54).

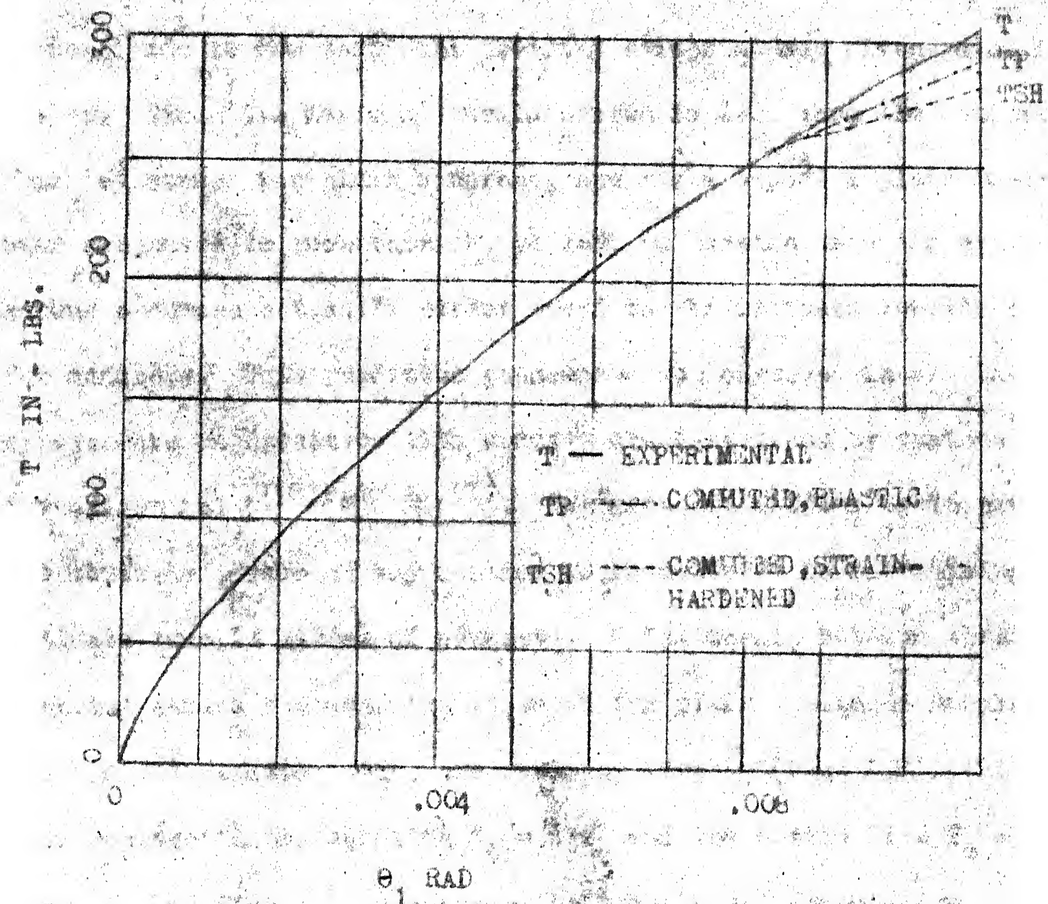


FIG. 2.5 Torque versus Rotation, T-Beams
(From Ref. 54).

(iii) Plastic theory

(iv) Non-linear stress-strain curve of concrete

(v) Hsu's Bending theory

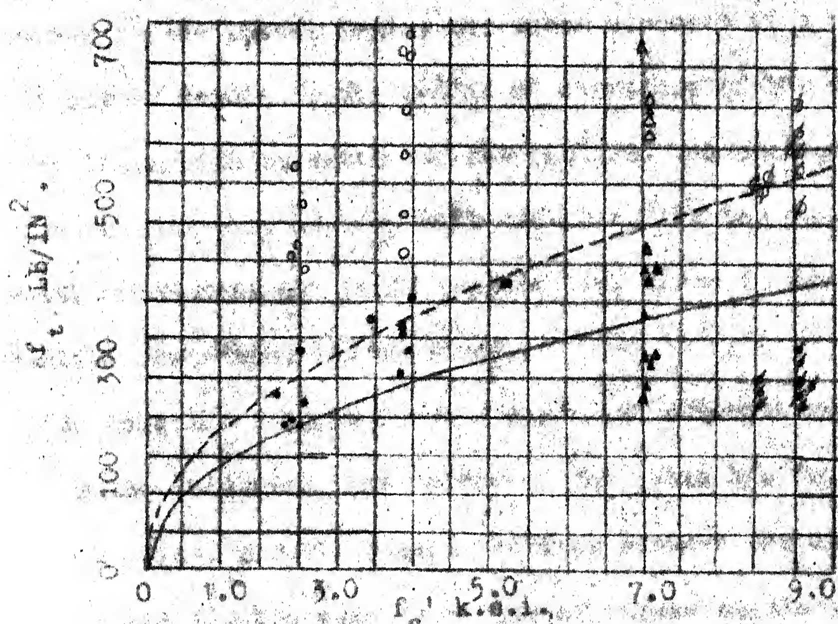
We shall discuss the various theories quoted above.

2.2.1 ELASTIC THEORY WITH BACH'S APPROXIMATION[17] :-

It can be shown that in a beam subjected to pure torsion, the principal stresses occur at angles of 45° w.r.t. the centre-line of twist and both the principal tensile and compressive stresses are equal in magnitude to the torsional shearing stress at any given point in the member. Since the maximum tensile stress is less than the maximum shearing stress for plain concrete, one would expect a plain concrete beam subjected to pure torsion, to fail in tension when the applied torque produces a tensile stress equal to the ultimate tensile stress for concrete. This predicted phenomenon was observed in all the experiments on specimens with concave cross-sections by various investigators[1,7,8,9,10,16,37]. The greatest difficulty in predicting the ultimate torque of any section is encountered in determining the ultimate tensile stress of concrete. Relationship between torsional shearing stress and concrete strength for plain specimens according to elastic and plastic theory are shown by Kemp[37] (Fig. 2.6). The full line represents the equation $f_t = 4\sqrt{f'_c}$ and the dotted line $f_t = 5.5\sqrt{f'_c}$ where f_t and f'_c are respectively the tensile and compressive cylinder strengths of concrete.

REMARKS :-

While the elastic theory gives the correct torsional resistance moments at working loads, it does not give the correct value for the



LEGEND

●	○	Nylander
■	□	Cowan
▲	△	Humphreys
✦	✧	Zia

NOTE : Solid points for plastic theory;
Open points for elastic theory.

FIG. 2.6 Relationship between Torsional shearing stress and Concrete Strength according to elastic and plastic theories
(From Ref. 37).

maximum shear stress. A re-entrant corner results in high stress over a small area near the corner even at low loads. Since the elastic theory is based on the assumption that stress is proportional to strain, the theory leads to the conclusion that at moderate loads, the maximum torsional shear stress is far beyond the ultimate strength of the material. For a T-section with a perfectly sharp re-entrant corner the elastic theory predicts infinitely high torsional shear stresses at the corner at the lowest loads; the shear stresses at a rounded re-entrant corner depend on the radius of curvature of the corner. In practice these high stresses are not realised because materials cease to behave elastically at very high stresses. In the case of the concrete, which does not yield plastically, there is nevertheless a substantial redistribution of stress.

An accurate formula for the torsional strength must therefore allow for the relatively low values of the ratio h/b for the web of the concrete section, for the junction-effect between the component rectangles and for the redistribution of stress at the re-entrant corners.
Here h = depth of the rectangular section ;
and b = breadth of the rectangular section.

For specimens with convex-sections, consistent results can be expected from elastic theory with suitable adjustment of the load factors used for design purposes. The elastic method when applied to specimens with concave cross-sections, such as T and I sections, predicts strengths which bear little relationship to experimentally measured values.

2.2.2 SAINT VENANT'S ELASTIC THEORY TAKING INTO ACCOUNT THE JUNCTION EFFECT[46,49] (T, L and I sections)

An approximate method to determine the torsional rigidity, has been given by Bach for T, I and L section by using narrow-rectangle formula and the torsional rigidity of the section is approximately given by the sum of the torsional rigidities of these component rectangles framing the section. The formula thus obtained gives good results when the breadth of the rectangles forming the section is very small when compared to their depth. But in concrete structures, this assumption is rarely satisfied.

To take into account the junction-effect, K.T.S. Iyenger[49] has evolved a rigorous method in which continuity conditions for the elastic stress functions are used at the junction of the two component rectangles framing the T or L section. For example, a T-section consists of two rectangles i.e. a flange-rectangle and a web rectangle and we name the stress function respectively for these two component rectangles as ϕ_1 and ϕ_2. Then using Saint Venant's theory of elastic torsion, in addition to boundary conditions on these component stress functions, we have the continuity conditions as :

$$\phi_1 = \phi_2 \text{ and } \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \quad \text{at the junction.}$$

Here (x,y) are the rectangular co-ordinates of a point of the cross-section. Then a solution using Fourier's series is obtained for this stress function ϕ of the whole section using these two above-said additional conditions.

DISCUSSION :-

The discrepency which creeps in while using narrow rectangular formula for T, I and L section is quite appreciable when compared with the solution obtained by the above-said rigorous analysis. For comparison, we give below the difference between the exact and the approximate analysis, as reported by Iyenger(49) :

T-SECTION : (Fig. 2.7)

The torsional rigidity 'C' of the T-section is given by :

$$C = \frac{2}{\theta} \iint \phi_1 \, dx \, dy + \frac{2}{\theta} \iint \phi_2 \, dx \, dy$$

Quantity	Exact Analysis	Bach's approx.	%age error
$\frac{C}{Gt^4}$	20.702	24.167	16.7
$\frac{\tau_{max}}{G\theta t}$	2.345	2.500	6.7

Here the particular T-section dimensions taken are :

$B = 5t$; $B_1 = \frac{1}{4} B$; $h = 4t$

θ = angle of twist per unit length

C = torsional rigidity defined as $= \frac{T}{\theta}$

T = applied torque

$\tau_{maximum}$ = maximum torsional shear stress

ϕ_1 = elastic stress-function for the region (1) shown in the figure (2.7)

ϕ_2 = elastic stress-function for the region (2) shown in the figure (2.7)

t = thickness of the flange

B = half flange-width

B_1 = half-web-width

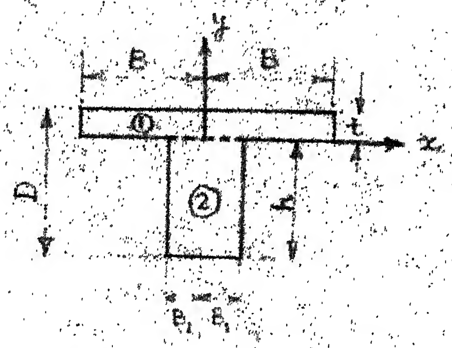


FIG. 2.7 T-Section (From Ref. 49)

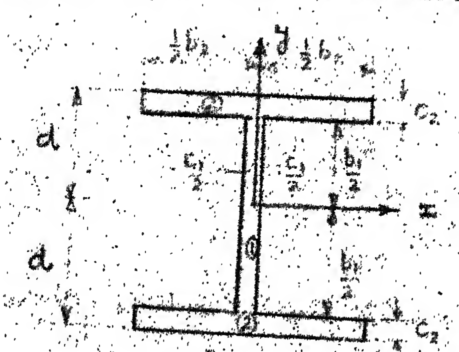


FIG. 2.8 I-Section (From Ref. 49)

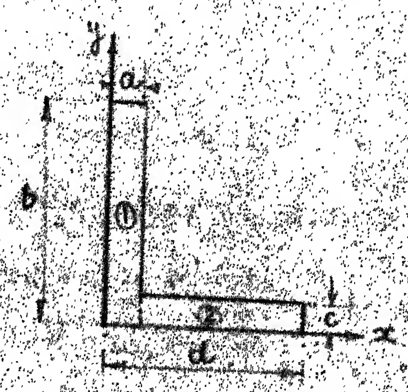


FIG. 2.9 L-Section (From Ref. 49)

h = depth of the web between bottom of the flange and bottom face of the web.

G = modolous of Elasticity in shear.

I - SECTION :-

Similarly for I section, comparison of the result is given below for the quantity $\frac{C}{Gc_2^2}$ (Fig: 2.8)

Two cases are considered : (i) $d = b_2$; $c_2 = 2c_1$; $b_2 = 5c_2$

and (ii) $d = b_2$; $c_2 = 2c_1$; $b_2 = 10c_2$

$\frac{b_2}{c_2}$	Exact analysis	Narrow rectangle formula	%age error
5	4.075	3.667	10
10	7.529	7.417	1.5

Here d = half of the over-all depth of the section

b_1 = depth of web between bottom of the top flange and top of the bottom flange

c_1 = thickness of the web

c_2 = thickness of the flanges

L - SECTION :- (Fig. 2.9) Comparison of the Results for the quantity C/Gc^4 is given below; two cases are taken :

(i) $b = d$, $a = c$, $b = 5c$

(ii) $b = d$, $a = c$, $b = 10c$

$\frac{b}{c}$	Exact analysis	Narrow Rectangular formula	%age error
5	3.275	3.000	8.4
10	6.405	6.333	1.12

Here b = total depth of the vertical leg; a = width of the vertical leg;

d = total width of the horizontal leg; c = depth of the horizontal leg;

2.2.3 PLASTIC THEORY (13,14,15)

Any prismatic bar undergoes plastic deformation in certain parts of its cross-section when it is twisted severely. An idealised stress-strain diagram with a flat plateau may be assumed and as long as the shearing strains remain quite small, the states of stress in the bar subjected to pure torsion are taken as states of simple shear only. The state of simple shear has two components τ_{xz} and τ_{yz} whose resultants has a constant value ρ in those parts of the cross-section where the plastic state of stress exists. Therefore, by the condition of plasticity, τ_{xz} and τ_{yz} should satisfy :

$$\tau_{xz}^2 + \tau_{yz}^2 = \rho^2 = \text{constant}$$

and also : $\dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} = 0$

Here τ_{xz} = x-component of the torsional shear stress

and τ_{yz} = y-component of the torsional shear stress.

This equation is satisfied by putting $\tau_{xz} = \dfrac{\partial F}{\partial y}$ and $\tau_{yz} = -\dfrac{\partial F}{\partial x}$ where

F(x,y) is the plastic stress function of the cross-section.

So we get :

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 = \rho^2$$

or / grad F / = ρ = constant slope and hence the sand - heap analogy. Plastic twisting moments for various cross-sections[49], using Sand - Heap analogy are given below :- (The have been obtained by Iyenger) :

T - SECTION : (Fig : 2.7)

$$V = \left[\frac{4}{3} B_1^3 + B_1^2 (D - 2B_1)\right]\rho + 2\left[\frac{t^3}{6} + \frac{t^2}{4}(B - B_1 - t) + \frac{t^3}{12}\right]\rho$$

where V = volume under the sand-heap.

and $T_{pl} = 2V$ where T_{pl} = Plastic torsional capacity

L - SECTION : (Fig.: 2.9)

$$V = \left[\frac{a^3}{6} + \frac{a^2}{4} (b - a) + \frac{c^2}{4} (d - a)\right]\rho$$

$$T_{pl} = 2V$$

I - SECTION : (Fig. : 2.8)

$$V = 2\rho\left[\frac{c_2^3}{6} + \frac{c_2^2}{4} (b_2 - c_2)\right] + \left[\frac{c_1^2}{2} (d - c_2) + \frac{c_1^3}{6}\right]\rho$$

$$T_{pl} = 2V.$$

RECTANGULAR SECTION

$$V = \tfrac{1}{2} b^2 (h - 1/3\, b)\rho$$

$$T_{pl} = 2V.$$

DISCUSSION :

The experimental evidence due to Nylander(13,14) suggests that plastic theory of torsion can be applied to concrete T and L-beams. This results in much simpler formulae and in some economy of the material.

2.2.4 NON-LINEAR STRESS-STRAIN CURVE OF CONCRETE :

The following analysis is due to G.R. Buchanan(54). Due to the warping of the non-circular cross-sections, the strain variation would not necessarily be linear. Also stress and strain does not have

a linear relationship and the shearing modulous G in the Poisson's equation for torsion i.e.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\frac{\theta_1}{L}$$ where ϕ = elastic torsional stress-function and θ_1 = total angle of twist in a length L,

is not a constant but varies as the stress changes. So G is assumed a function of co-ordinates location and we get instead of the above, the following equation :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2\frac{\theta_1}{L} G(x,y)$$

This equation is then solved by the method of Finite Differences, i.e.

$$-4\phi_{i,j} + \phi_{i,j+1} + \phi_{i,j-1} + \phi_{i+1,j} + \phi_{i-1,j} = W$$

where $W = -2\frac{\theta_1}{L} G(x,y)\, h^2$ (Fig. 2.10)

and h = distance between grid-points.

A difference equation is written for each grid point resulting in a system of 'n' equations in 'n' unknows. The constant W is calculated for each grid point and the resulting equations are solved to obtain the stress-function by inverting the matrix of the coefficients and multiplying by the solution vector of W-values.

Now, $\tau_{xz} = \frac{\partial \phi}{\partial y}$ and $\tau_{yz} = -\frac{\partial \phi}{\partial x}$

Therefore to get the component shearing stresses, it is necessary to differentiate the stress-function obtained above for each grid point. This is done numerically by using difference-tables and interpolation

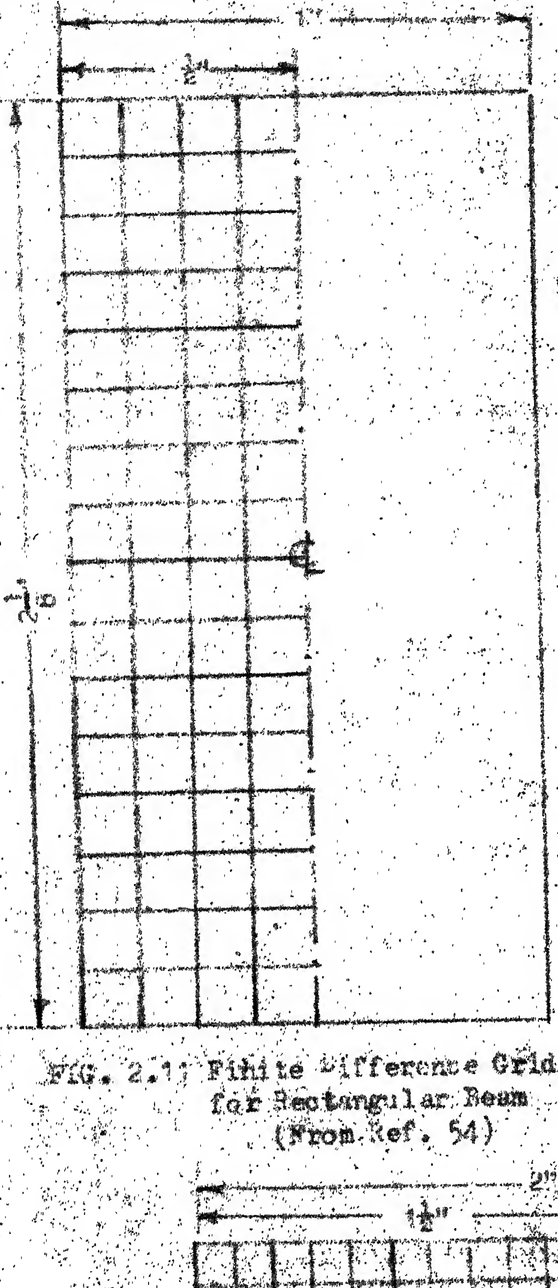


FIG. 2.11 Finite Difference Grid for Rectangular Beam (From Ref. 54)

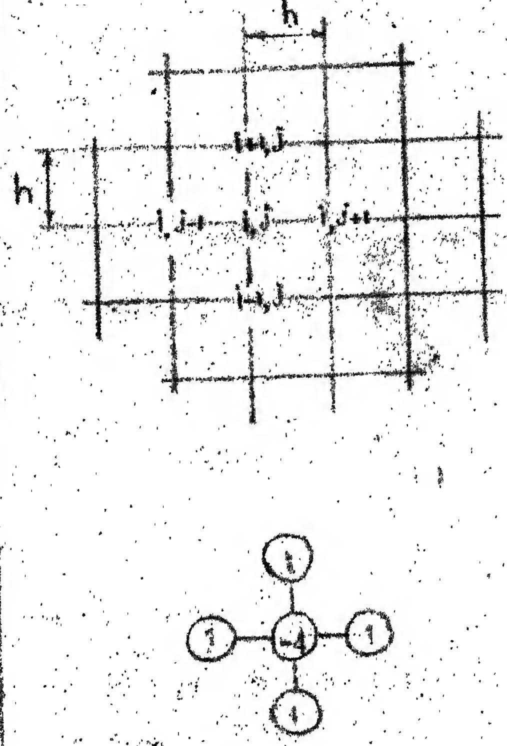


FIG. 2.10 Grid and Difference Operator for Poisson's equation (From Ref. 54)

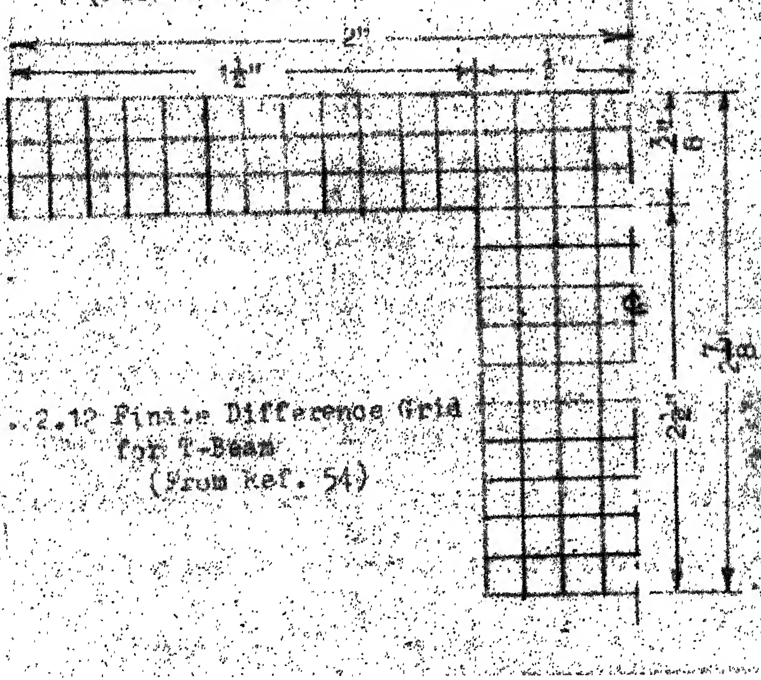


FIG. 2.12 Finite Difference Grid for T-Beam (From Ref. 54)

formulae. After determining the stresses in both directions at each grid point, the resultant shear stress at each point is calculated as :

$$\tau = \sqrt{(\tau_{xz})^2 + (\tau_{yz})^2}$$

The relationship between shearing modulous and shearing stress is obtained from the experimental relationship between torque and rotation for a circular beam. The shearing stress τ is approximately given by :

$$\tau = \frac{T\ r}{J}$$

where T = The applied torque

r = The radius of the test specimen

J = The polar moment of Inertia of the test specimen and the shearing strain is given by :

$$\gamma = \frac{\theta_1\ r}{L}$$

Then G = slope of the $(\tau - \gamma)$ curve. The slope of shear stress-shear strain $(\tau - \gamma)$ curve is plotted against the shearing stress. The method of least squares is used to obtain an equation for γ as a function of τ from the experimental data. The first derivative gives the desired relation between $G(x,y)$ and τ.

The analytical process for determining the torque - capacity stress-distribution, and theoretical torque-rotation $(T - \theta_1)$ curve for each beam is done as given below :

(i) A suitable finite-difference grid is chosen for the beam cross-section. The difference equation is written for each grid-point and the matrix of coefficients inverted.

(ii) A very small value for rotation θ_1 is assumed, (say 1×10^{-4} radians) and a large value for shearing modulous 'G' is assumed, (say 2×10^6 p.s. i.) and for this initial value of 'G', it is assumed independent of the co-ordinate location of the point (x,y) to start with.

(iii) W is evaluated and then stress-function ϕ is found for each grid-point.

(iv) The shearing stress at each grid point is then calculated.

(v) Shearing modulous is found corresponding to the above calculated shearing-stress, τ, from the shear modulous-torsional shear stress ($G - \tau$) curve.

(vi) θ_1 is then increased by a small amount and a new W is found based upon new θ_1 and the above calculated 'G' and a new value of stress-function, ϕ obtained.

(vii) Steps (iv to vi) are then repeated until a limiting experimental value of rotation is attained.

(viii) The torque is calculated after every five increments of rotation from : $T = 2 \iint \phi \, d A$.

These calculations yield a plot of ($T - \theta_1$) as well as the stress distribution over the cross-section at increasing increments of rotation. We give below the results of the application of the above theory for different shapes of cross-section.

CIRCULAR BEAMS :- (Buchanan's Tests Ref. 54) :

Three circular beams C-1, C-2 and C-3 (All plaster models) were tested in pure torsion to obtain experimental ($T - \theta_1$) data. Shearing stress and shearing strain curves were computed from the

experimental ($T - \theta_1$) curves (Fig.: 2.1)

RECTANGULAR BEAMS :- (Buchanan's Tests Ref. 54) :

One rectangular specimen 1" x $2\frac{1}{8}$" was tested in pure torsion. The experimental ($T - \theta_1$) curves are shown (Fig. : 2.2)

For the analytical solution the cross-section was divided in half abut its vertical axis of symmetry. 1/8" grid was used (Fig. : 2.11)

The ($T - \theta_1$) curve and stress distribution for the rectangular beam were computed using two different ultimate stress distribution theories :

(a) It was assumed that the shear stress would reach a plastic condition and ;

(b) it was assumed that the stresses would redistribute in a predetermined strain hardened manner (Figs.: 2.3, 2.4)

T - BEAMS : - (Buchanan's Tests Ref. 54) :

One T - shaped specimen as shown, was tested in pure torsion to obtain experimentally ($T - \theta_1$) data. The experimental ($T - \theta_1$) curve is shown (Fig. 2.5)

For the theoretical solution the cross-section was divided in half about its vertical axis of symmetry and 1/8" grid was used.(Fig.: 2.12)

Two stress distributions were obtained by assuming both a plastic stress distribution and a strain - hardening stress distribution. (Figs.: 2.13, 2.14)

DISCUSSION :-

Buchanan computed ($T - \theta_1$) curves for the rectangular and T-beams and compared with existing data[8,12,18] in the literature. The

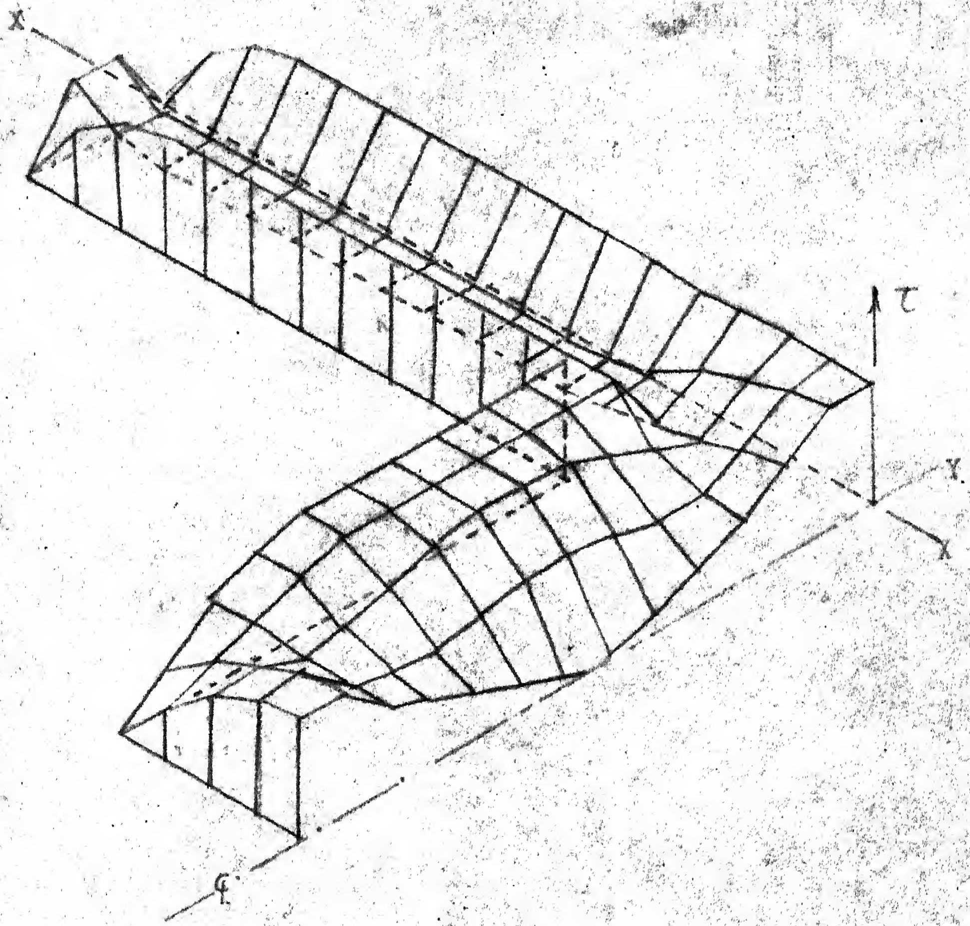


FIG. 2.13 Plot of Plastic Shearing Stress Distribution
for one half of the T-Cross-Section.
(From Ref. 54).

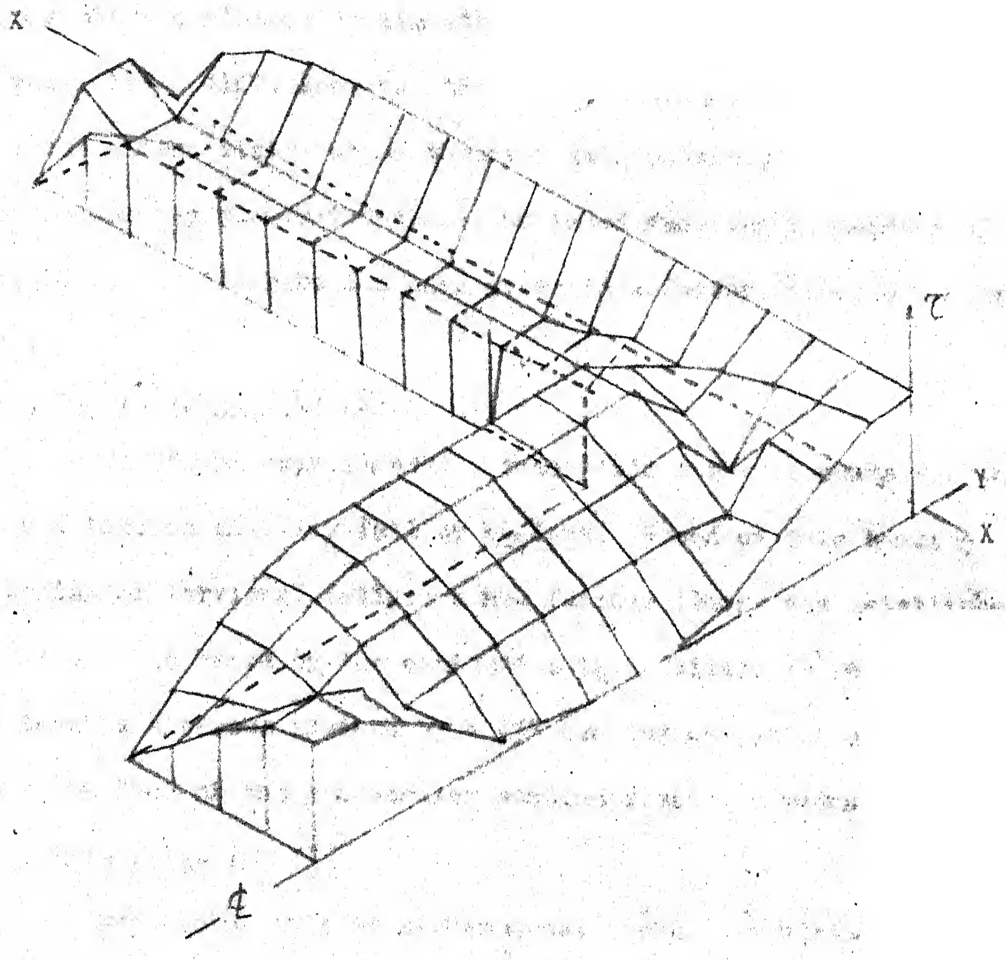


FIG. 2.14 Plot of Strain hardened Shearing Stress Distribution for one half of the T-Cross Section (From Ref. 54).

comparison has been shown (Figs.: 2.15, 2.16)

Previous investigators[13,14,53] have suggested that a material such as structural concrete exhibits a plastic stress distribution near failure. It appears that a strain - hardened stress distribution would be more reasonable for a material that does not have a plastic plateau on its stress-strain curve. There appears to be very little difference in the computed torque capacity for either ultimate stress distribution theory. This possibly explains why torque capacity prediction-equations based upon the assumption of plastic stress distribution have given satisfactory results in the past.

2.2.5 HSU'S THEORY :[57,58]

Hsu found experimentally that plain concrete beams subjected to pure torsion actually fail by <u>bending</u>. Based on this bending mechanism of torsional failure a new failure theory was established. This theory is based on the contention that failure is reached when the tensile stresses induced by a 45° bending component of torque on the wider face of the rectangular section reaches a reduced modulous of rupture (Fig. : 2.17)

Expressing this in mathematical terms, the ultimate torque, T_{up}, (in pound-inch) is given by :

$$T_{up} = \frac{b^2 h}{3} (0.85 f_r)$$

where T_{up} = ultimate torque capacity of plain concrete specimen

b = breadth of the rectangular section

h = depth of the rectangular section

f_r = modulous of rupture of concrete.

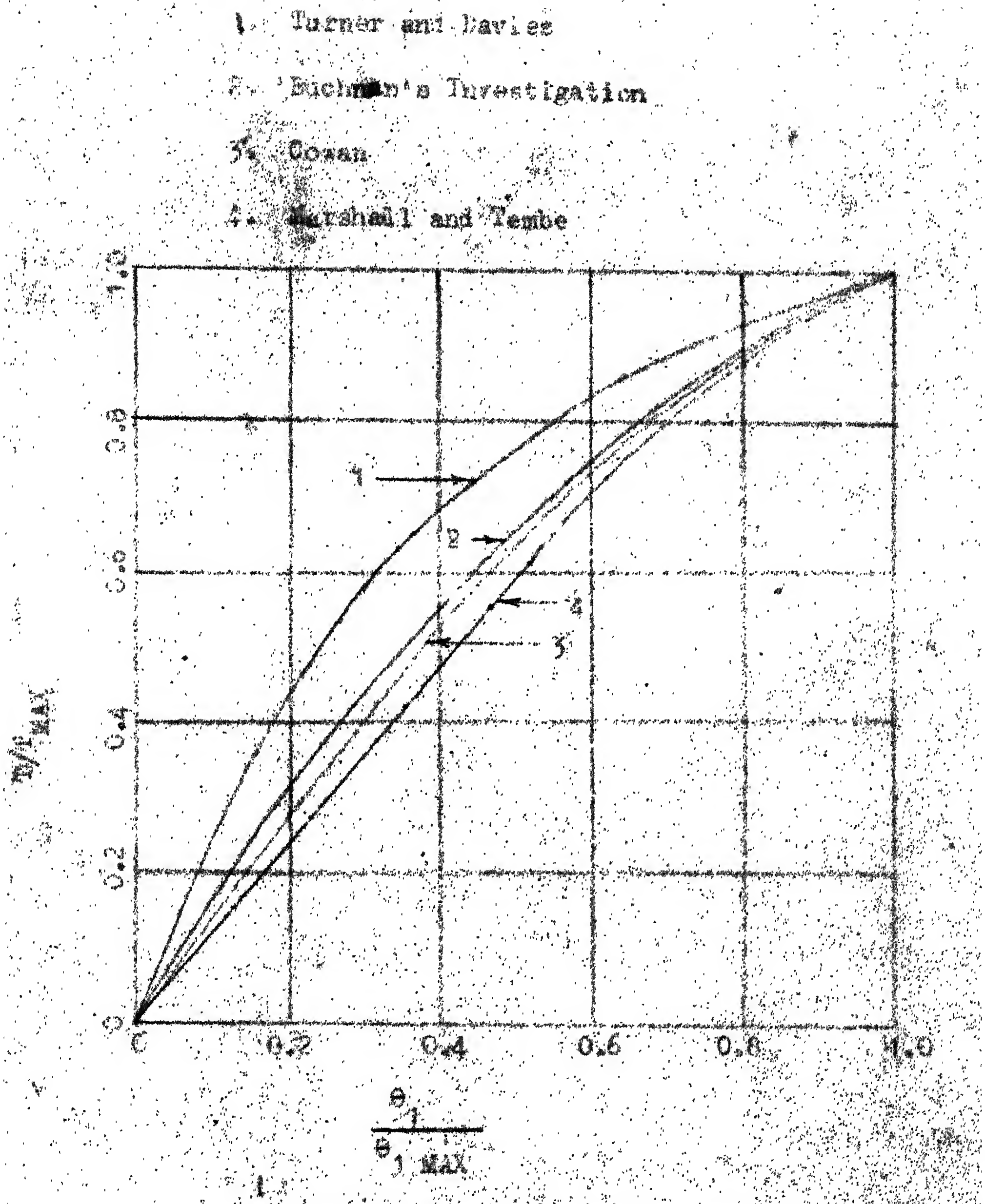


FIG. 2.15 Non-Dimensional Plot of Torque versus Rotation for Rectangular Beams
(From Ref. 54).

1. Marshall and Tembe

2. Turner and Davies

3. Buchanan

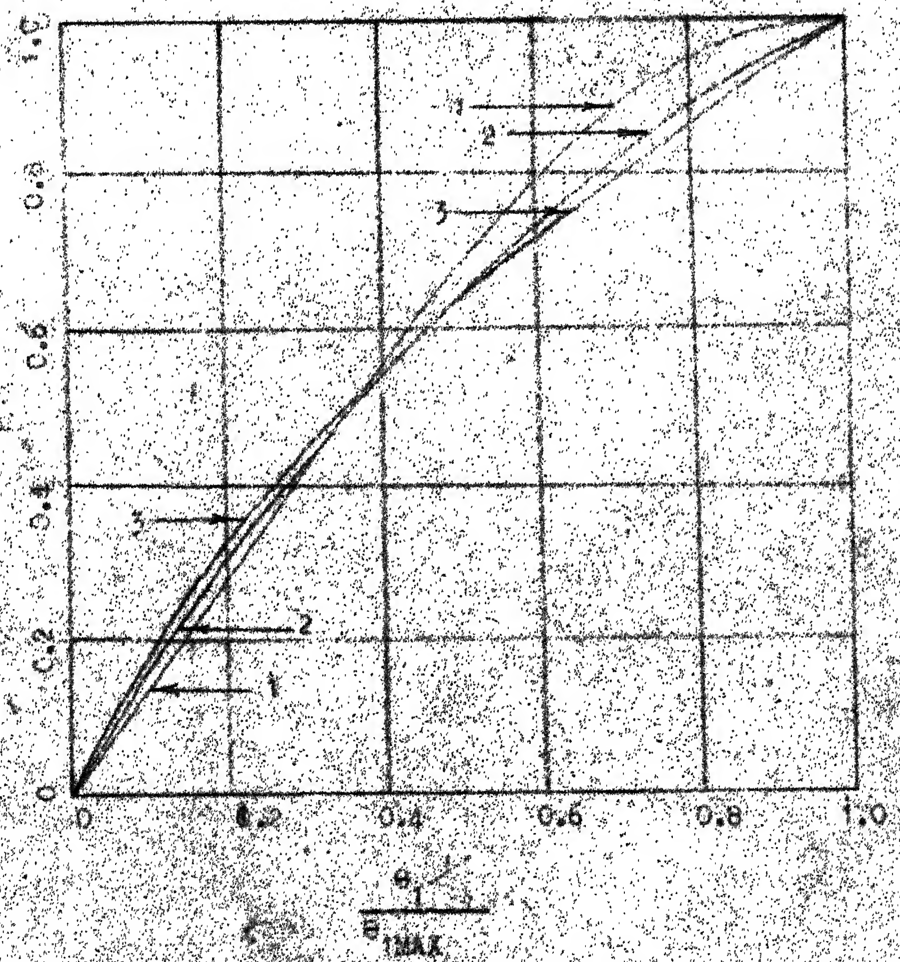
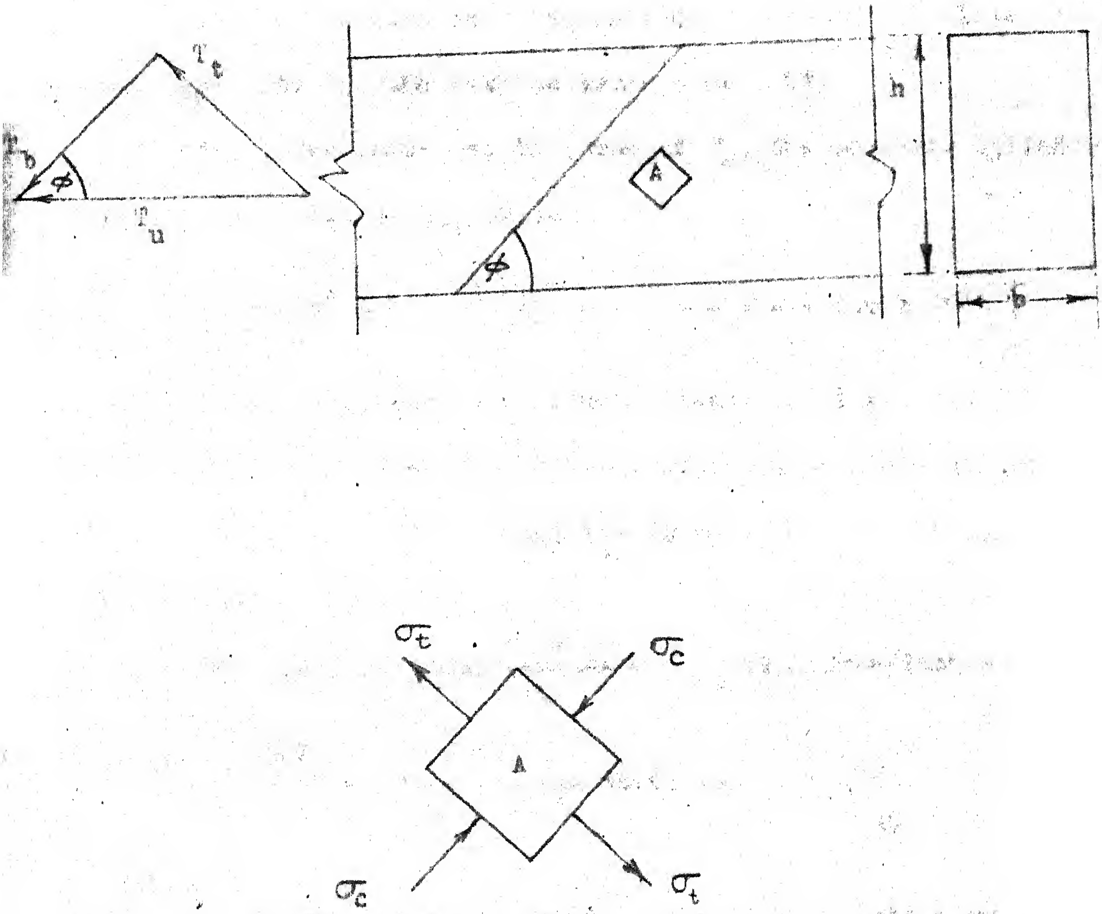


FIG. 2.16 Non-Dimensional Plot of Torque versus Rotation for T-Beams (From Ref. 54).



σ_c = Principal Compressive Stress

σ_t = Principal Tensile Stress

FIG. 2.17 Bending Component (T_b) and Torsional Component(T_t) on a Beam (From Ref. 58).

This equation was checked by Hsu[57,58] at the Portland Cement Association Laboratories, Chicago, by conducting tests on 10 beams with various cross-section. The degree of agreement that was found is indicated by the fact that the ratios of test to calculated torque varied from 0.94 to 1.06 with an average of 1.01.

The above equation, in terms of f'_c, the concrete cylinder strength, was given by Hsu as :-

$$T_{up} = 6(b^2 + 10)\, h^3 \sqrt{f'_c} \quad \text{- - - - - - - - - -} \quad \text{for } b > 4'' ,$$

where f'_c is the compressive cylinder strength of concrete. This expression was compared with 55 beams reported by eight groups of authors[2,4,10,13,18,26,60] and was found to be in good agreement by Hsu with the test results.

The angle of twist $\dfrac{(\theta_1)_{up}}{L} = \theta_{up}$ in degree/inch was found to be : $\theta_{up} = \dfrac{0.025}{\sqrt{bh}} \left(1 + \dfrac{10}{b^2}\right)$, and stiffness $C_t = \dfrac{T_{up}}{\theta_{up}}$

where θ_{up} = angle of twist per unit length for plain concrete specimen at failure.

T AND L-SECTIONS : The following observations have been made by Hsu :-

Although beams of T and L sections subjected to pure torsion also fail by bending the failure torque is very difficult to calculate based on bending mechanism. For design purposes it is convenient to divide a T or L section into stem and flange sections each of which is rectangular. The strength of the entire section of the beam is taken as the sum of the strengths of stem and flanges calculated by the above equation. This results in the following expression :

$$T_{up} = \sum 6(b^2 + 10) h^3 \sqrt{f'_c}$$

REMARKS :

Two categories of theories for calculating the torsional strength of a plain concrete rectangular beam have been proposed. These are the elastic theory and the plastic theory. In elastic theory, stresses in a beam are distributed according to Saint Venant's theory. In plastic theory, concrete is assumed to be a plastic material. These theory employ the maximum tensile stress theory to predict the failure torque of a beam. In each case it is assumed that failure occurs when the maximum principal tensile stress becomes equal to the tensile strength of the concrete.

All these theories are less than satisfactory. The elastic theory greatly under-estimates the torsional strength of a rectangular beam. The sand-heap analogy although able to account roughly for the excessive strength is theoretically unsound because concrete is known to have little plastic behaviour, especially in tension.

These difficulties indicate that the maximum tensile stress assumption which is the basis of the above-said theories, may be incorrect, as observed by Hsu. Therefore, the basic mechanism of failure re-examined under bending theory by Hsu is promising and more research is needed in this direction.

3. REINFORCED CONCRETE MEMBERS WITH EITHER TRANSVERSE OR LONGITUDINAL REINFORCEMENT IN PURE TORSION

3.1 BEHAVIOUR[37,46] (Tests by Kemp and Cowan) :

The presence of reinforcement in one direction only has little effect on the behaviour of a specimen as compared to an equivalent plain concrete specimen. As in the case of plain concrete specimens, a specimen reinforced in one direction only fails suddenly with the development of the first 45° helical crack. As the ultimate torque is reached, there is a certain amount of in-elastic redistribution of stresses as evidenced by the ($T - \theta$) curves which bend over just before failure. Unlike the plain specimens, the sections with longitudinal steel only exhibit a small amount of ductility after the ultimate torque has been reached.

Those specimens reinforced with hoops only displayed no ductility once the ultimate torque had been reached. The failure was sudden and destructive. It was observed in general that the presence of either longitudinal or hoop reinforcement only has very little effect on the strength or stiffness of the specimen.

3.2 STRENGTH[37,46]

Kemp[37] has shown that in the case of specimens reinforced in one direction only, the steel is unable to provide a force component in a direction at 45° to the axis of twist. When the principal tensile stress reaches the tensile capacity of the concrete, a crack forms and the member fails. Thus the steel contributes very little to the torsional capacity of the member.

3.3 SPIRAL AND HOOP REINFORCEMENT[37,46] :-

Substantial increase in torsional strength according to Kemp[37] and Cowan[46] can be achieved only by providing reinforcement in the direction of the principal tensile stresses. If concrete is twisted, it fails along a spiral line at 45° to the axis when the maximum diagonal tensile stress exceeds the tensile strength of the material. The most effective reinforcement consists of a series of 45° spirals. Spiral reinforcement has, however, two disadvantages: rectangular spirals are not easily made and left-hand and right-hand spirals are required in beams subject to a reversal of twisting moment.

4. REINFORCED CONCRETE MEMBERS WITH BOTH LONGITUDINAL STEEL AND TRANSVERSE STEEL REINFORCEMENT, IN PURE TORSION (REFERENCES : 5,6,8,11,12,46,50,57)

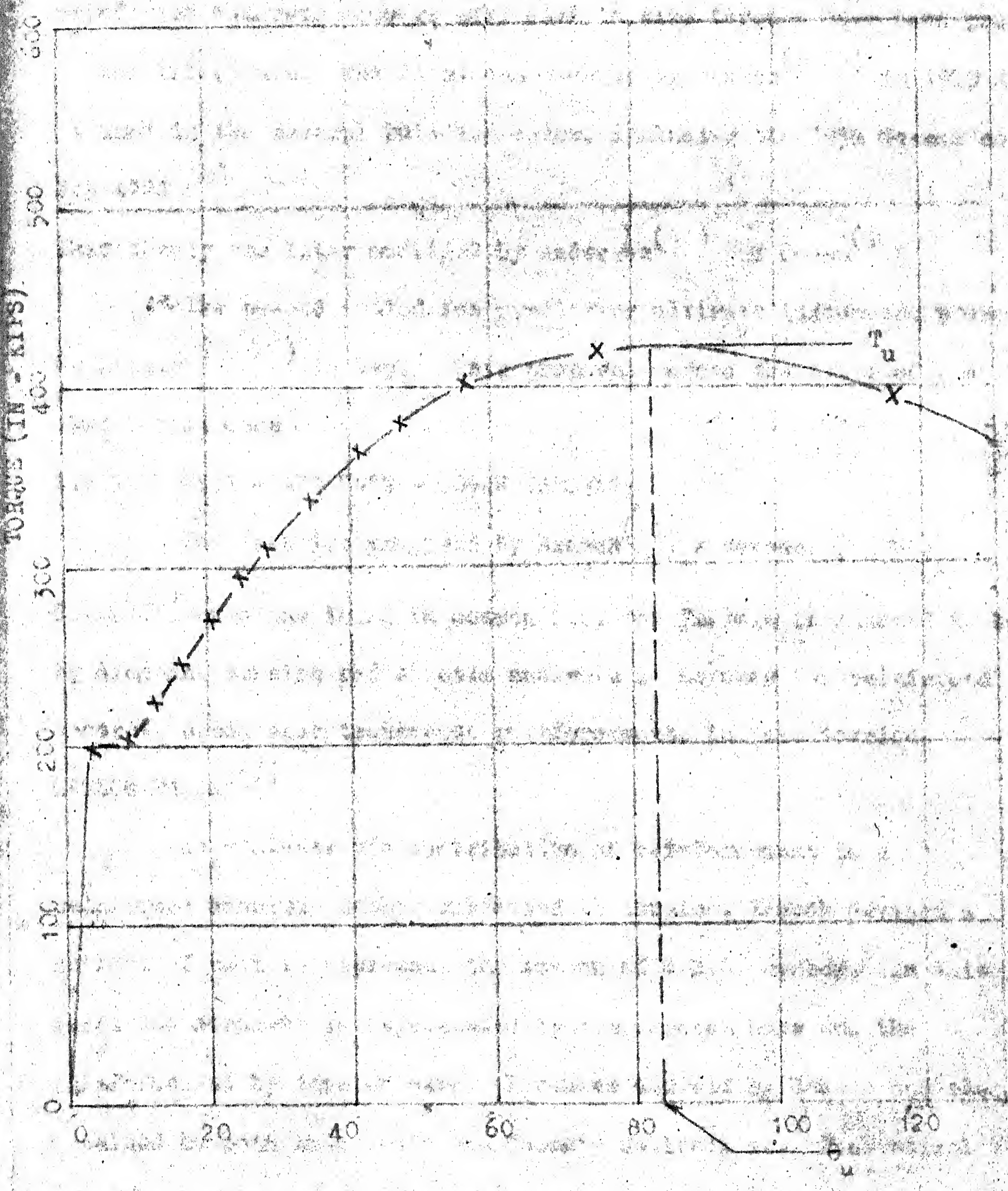
4.1 BEHAVIOUR : (Based on, as reported in the above references) :

The behaviour of any type of specimen upto cracking was nearly the same regardless of whether it was plain, reinforced longitudinally or transversally, or continuously bound by a helix or hoops and longitudinal bars. The specimens behaved elastically upto cracking and the stiffness and cracking torque appear to depend almost entirely on the geometry of the cross-section and the concrete strength and very little on the amount or disposition of any reinforcement present.

After the cracking torque was exceeded the reinforced specimens continued to gain strength although loosing stiffness until the ultimate torque was reached. The $(T - \theta_1)$ curve showed much of ductility.

When the applied twisting moment reaches a certain magnitude, the beam cracks and rotates under a constant torque while the stress in the reinforcement increase suddenly. After this twisting has occured, the load again increases. The torsional stiffness in this third portion of the $(T - \theta_1)$ curve is approximately a constant and steel stresses increase linearly with the torque. When the ultimate torque is approached the curve turns gradually towards the horizontal (Fig.: 4.1)

As T_u, the ultimate torque-capacity for a reinforced section, is always found to be more than T_{up}, this increase being only due to the presence of reinforcement.



ANGLE OF TWIST (10^{-3} DEG./IN.)

FIG. 4.1 Typical Torque-Twist Curve for a Rectangular R.C.C.Section (From Ref. 57).

4.2 STRENGTH :

Two methods for predicting the ultimate torque of rectangular reinforced concrete members subjected to pure torsion have been proposed in the literature. The first was developed by Rausch(6,7) in 1929 and is used in the several building codes, including the 1958 German code DIN 4224(68).
This theory was later modified by Andersen(11) and Cowan(46)

The second method for predicting ultimate torque was proposed by Lessig(29-31) in 1958. This proposal formed the basis of the 1962 Soviet Code(67).

4.2.1 RAUSCH - ANDERSEN - COWAN THEORIES

The theories proposed by Rausch(7), Andersen, (9,10), Cowan(16) have one thing in common i.e. the failure is assumed to be by diagonal tension and elastic analysis is assumed for reinforced concrete beams with transverse reinforcement, in pure torsion.

RAUSCH THEORY(7)

To evaluate the contribution of reinforcement in a reinforced concrete member subjected to torsion, Rausch devised a network of bars to represent the action of a R.C. member. In this model the concrete is represented by compression bars and the reinforcement by tension bars. Formulae derived by Rausch can also be obtained by both Andersen's and Cowan's derivations. Theoretical solutions on the basis of elastic theory were put forward by Rausch and Andersen. Rausch's Theory is based on the same assumptions as are laid down in the British standard code of Practice (C.P. 114) :

1. Both steel and concrete are elastic within the range of permissible stresses and

2. All the tensile stresses are taken by the reinforcement except that the concrete may be assumed to resist diagonal tension within the limits of shear stress specified.

Andersen's Theory(9,10,11):

In accordance with the provisions of the American Code of Practice,(66) it provides that the reinforcement shall take only the tension in excess of that permissible on plain concrete.

Theories due both to Rausch and Andersen can be stated in the form :

$$T = \frac{\vartheta \sqrt{2} \quad A \times f_{st.p} \; \sigma_{st.p}}{u_p}$$

where T = the permissible torsional moment

A = the effective cross-sectional area of the section, defined as the area contained within the centre line of the spiral reinforcement

$f_{st.p}$ = the cross-sectional area of one spiral wire

$\sigma_{st.p}$ = the maximum permissible stress in the spiral reinforcement

u_p = the pitch of the spiral reinforcement measured parallel to the axis of the beam.

ϑ = a empirical constant

Discussion of Rausch and Andersen's theories :

In Rausch's formula the coefficient $\vartheta = 2$ for all shapes of cross-section. In Andersen's formula it is 2 for circular sections and approximately $\frac{4}{3}$ for square and rectangular sections.

Both formulae are based on the circular section theory of torsion, and therefore accurate only when applied to circular sections. In applying his theory to non-circular section, Rausch assumes that the stress

in the spiral reinforcement at any point is directly proportional to the distance of that point from the axis of the beam. For rectangular sections, the reverse is , in fact, true.

Andersen, assuming parabolic stress distribution in the reinforcement, obtains the radius of theequivalent circular section by applying the Bach formula.

COWAN'S THEORY(46) :

Cowan reasoned that the stresses and strains along the length of each spiral bar should vary from zero at the corner to a maximum at centre of each face according to Saint Venant's stress distribution. By integrating the energy along the spiral reinforcement(16) and equating it to one-half of the work done by the external torque, he was able to obtain the coefficient ν . This factor is a function of height to width ratio of the section only.

For practical purposes he proposes ν to be 0.8. With this modified value of ν , Cowan gave the following equation :

$$f_{st.p} = \frac{\sqrt{2}\ T\ u_p}{4\nu\,\sigma_{st.p}.b'\ h'}$$

where b' and h' are respectively the width and depth of the rectangular section enclosed within the rectangular stirrups.

In practice, spiral reinforcement is hardly even used in rectangular beams because of the expense of making it.

Vertical closed hoops and longitudinal bars are a more convenient form of torsional shear reinforcement. The diagonal forces due to torsion are then, resisted by the components of tensile forces in the hoops and the longitudinal steel in the direction of the diagonal tension,

while the diagonal compressive forces are resisted by the concrete. Consequently the area required for the vertical hoops :

$$f_{st.2} = \sqrt{2}\ f_{st.p}$$

Moreover, Cowan proposes an equal amount of longitudinal steel distributed equally in all the four corners of a rectangular beam. So we get :

$$(F_{s1} + F_{s2}) = 2\, f_{st.2} \frac{b' + h'}{u}$$

where $F_{s1} = F_{s2}$ = cross-sectional areas of longitudinal tensile and compressive reinforcement respectively.

and u = spacing of the rectangular stirrups.

DISCUSSION :

The above elastic theory assumes that the torsional resistance of a R.C. member is the sum of the resistance of a plain concrete member and that contributed by the reinforcement.

There are three divergent views of how concrete in a reinforced beam resists torsion : These are :

1. That concrete carries a torque equal to that of an unreinforced beam. This approach was adopted in Australian Code[65].
2. A second view holds that since concrete is badly cracked near ultimate load, its ability to carry torque is greatly reduced and should be assumed to be zero. This reasoning is reflected in the German Code[68].
3. The third approach provides a compromise between the first two by computing the carrying capacity of the concrete on the basis of the core. of the reinforced concrete beams. This third view was most widely accepted by torsion investigators in the 1960's.

The strength prediction based on the Rausch-Andersen-Cowan theory and the reasoning described above are compared as shown (Fig.: 4.2). with some of the test results reported in literature.[57] This comparison shows that both the German and Australian code overestimates the torsional strengths.

4.2.2 LESSIG-YUDIN THEORIES[29,30,31,36,39]:

A theory based on ultimate torsional strength was first proposed by Lessig in 1958 and a more detailed paper was published in 1959. Observing that cracks occur diagonally on the surface of a beam, the failure surface chosen is shown (Fig.: 4.3).

This surface is formed by a continuous diagonal crack on three faces and the straight line $\overline{AB}$ on the fourth face connecting its ends. A region close to $\overline{AB}$ is considered in compression and the steel in this region is neglected. All bars outside the compression region are assumed to be in tension and to be stressed to yield. Based on this failure surface, Lessig chose two equilibrium equations ; equilibrium of moment along the neutral axis x-x and the equilibrium of forces along the axis perpendicular to the compression zone. By minimising the moment equilibrium equation, it was found that a theoretical minimum torsional resistance occurs when the neutral axis x-x is parallel to either a longer face or a shorter face. This results in two modes of failure shown (Figs.: 4.4, 4.5).

DISCUSSION :

Lessig's theory considers both the combined resistance of concrete and reinforcement and the redistribution of stresses. Therefore, it explains four observed phenomena reported in the literature[50]. These phenomena can not be explained by the elastic theory :

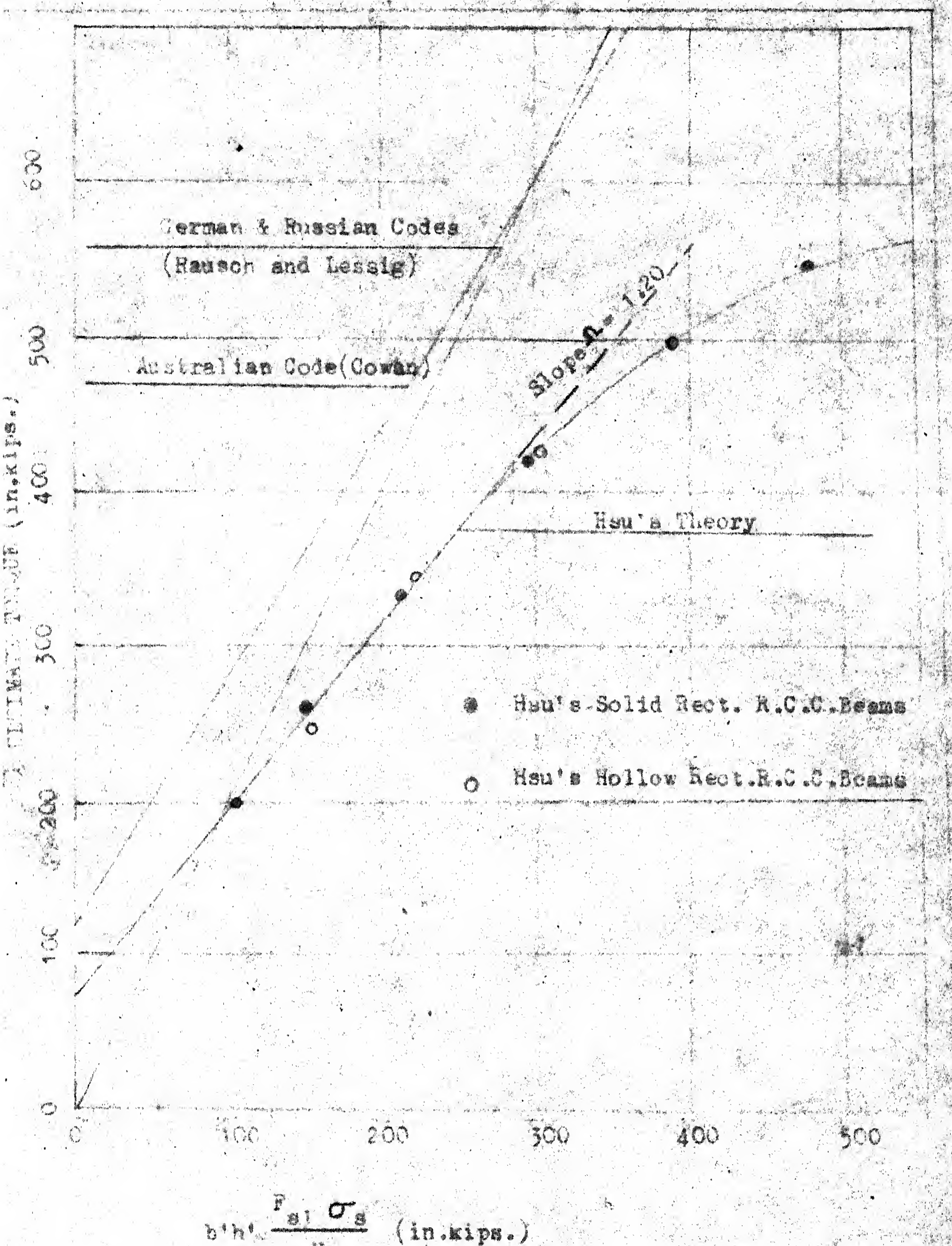


FIG. 4.2 Ultimate Torque for parameter $\frac{b'h'F_{s1}\sigma_s}{u}$ for Rect. R.C.C. Beams (From Ref. 57).

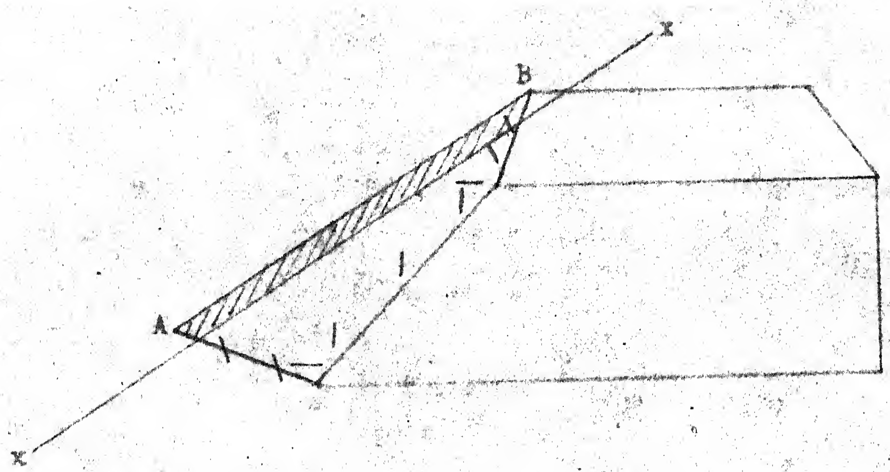
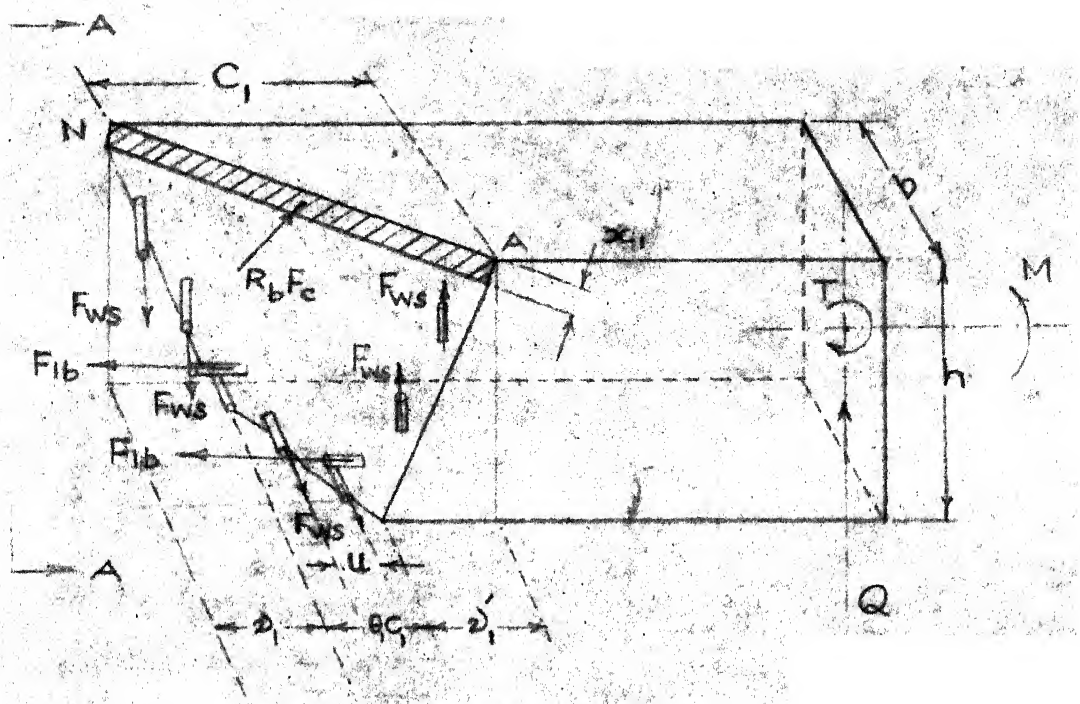
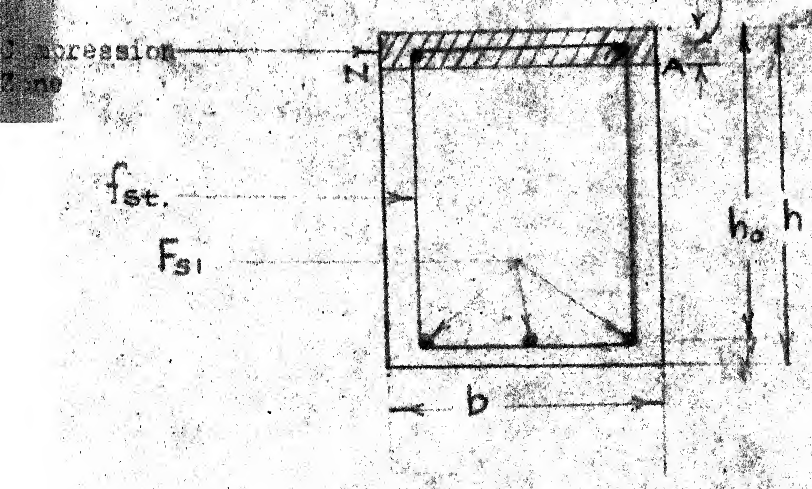


FIG. 4.3 Failure Surface of Free body according to Lessig.



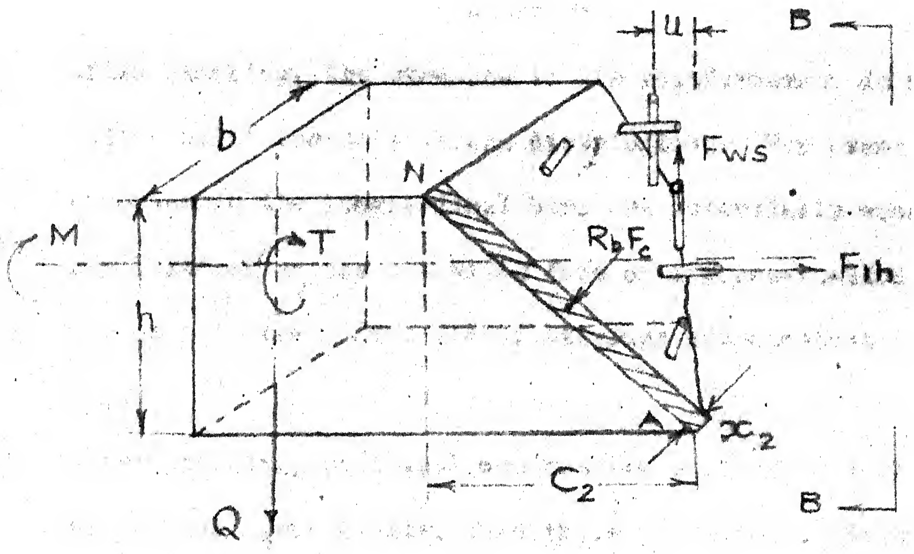
LONGITUDINAL ELEVATION



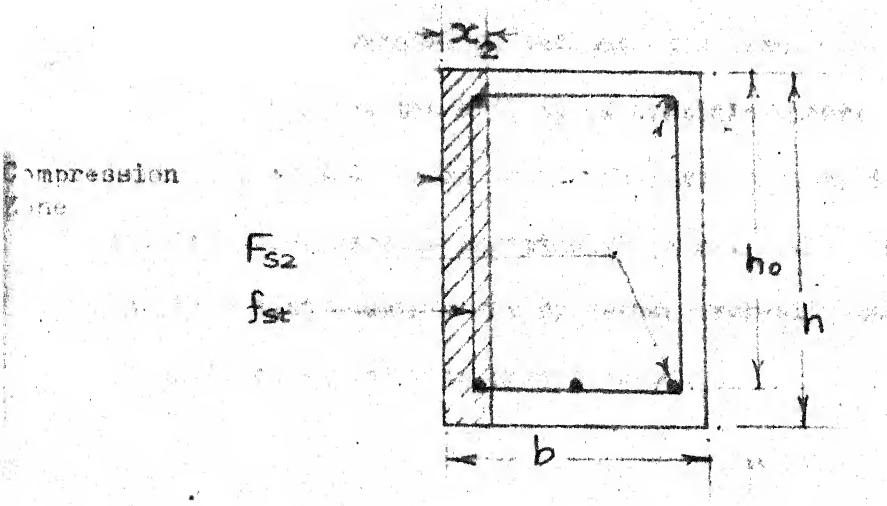
SIDE ELEVATION - AA

LESSIG'S FAILURE MODE 1.

FIG. 4.4 (From Ref. 79).



LONGITUDINAL ELEVATION



SIDE ELEVATION - BB

LESSIG'S FAILURE MODER 2

FIG. 4.5 (From Ref. 79).

1. At cracking, a reinforced concrete beam continues to twist under a constant torque. until the reinforcement is brought into action. This indicates a transition from Saint-Venant's equilibrium condition to a new one.

2. After cracking, the stresses in the reinforcement do not follow Saint Venant's stress distribution. For example, the stresses in the longitudinal bars are essentially equal for any location across the wider face of a cross-section. Also the stresses in the longer leg of stirrups are constant along that leg.

3. After cracking, principal compressive strains of concrete may be several times greater than those predicted by Saint Venant's theory.

4. The concrete core of a reinforced concrete beam does not contribute to the torsional resistance of a beam.

Despite the ability of Lessig's theory to explain the general behaviour of R.C. beams loaded in pure torsion, it rather approximately fits the test results numerically (Fig.: 4.2) From the figure, it is clear that Lessig's theory rather over-estimates the torsional strength of a member in pure torsion.

SECTION 5 : PRISMATIC R.C.C. SPECIMENS SUBJECTED TO COMBINED BENDING AND TORSION

5.1 SOLID T,L, AND RECTANGULAR CONCRETE SPECIMENS CONTAINING ONLY LONGITUDINAL REINFORCEMENT

5.1.1 INTRODUCTION :

Towards the understanding of the behaviour of concrete specimens containing both longitudinal and web reinforcement, under combined bending and torsion, a good study of the behaviour of concrete specimens containing longitudinal reinforcement only is necessary. In this area work of Nylander[13,14], Ramakrishnan[40], Gesund and Boston[42] for rectangular sections, and Victor and Ferguson[63,71] for T sections are available for study. In the experiments conducted by these researchers, the effect of variation of amount of steel, concrete strength, depth to width ratio of the cross-section and the ratio of applied torque to applied bending moment i.e. $K = T/M$, on the behaviour and strength, has been reported.

5.1.2 BEHAVIOUR :

Beams containing only longitudinal steel and loaded in combined flexure and torsion have been observed by various researchers[13,40,42] to fail in several modes, the mode depending in any particular case on the ratio of twisting moment to bending moment and the section properties of the member. Based on the observations of the above-cited researchers, the sequence of events comprising failure may be described as follows : Application of the bending moment cracks the lower portion of the beam and reduces the effective cross-section; further, it applies a compressive force to the uncracked concrete zone and a tensile force to the steel. When the member is twisted shearing stresses are induced in the uncracked concrete zone and dowel forces

on the steel. If spalling of the concrete is ignored failure of the beam will be initiated either by the concrete failing under the combined compressive and shearing stresses or by the steel yielding due to the tensile and dowel forces. The failure is sudden in case of predominant torsional loading while for the other extreme if the section is under-reinforced the failure will be gradual.

5.1.3 STRENGTH :

Ramakrishnan(40), reports the following points regarding the strength of beams without web reinforcement, based on his experimental tests :

(i) The ultimate strength of such beams depends mainly on the tensile strength of concrete.

(ii) The plastic analysis due to Nadai(15) has proved to be valuable in predicting the ultimate strength of these beams.

(iii) The effect of twisting moments is to reduce the bending resistance, while the addition of bending moment increases the torque capacity of such beams. But there is a limit beyond which the addition to more bending moment reduces the torque capacity of beams and the reduction in the bending resistance of the beam is practically nil. The limiting value of $\frac{M}{T} = 4$ to 6 depending on many variables.

Hsu's Approach(56,72) :

Hsu has developed a 3-dimensional interaction surface (Fig. 6.1) for predicting the ultimate strength in the case of beams without web reinforcement, subjected to combined bending, torsion

and shear. The interaction curve between applied moment and torque is then the special case of Hsu's interaction surface, when shear force is zero and so the problem becomes two dimensional. However, this interaction surface, shall be discussed in sub-section 6. (pp:)

5.1.4 REMARKS :

Hsu's simplified interaction surface (Fig. 6.1) appears promising in predicting the strength of such members especially for design since the ultimate equilibrium theories of Lessig (discussed in chapter 6) fail when there is no web reinforcement.

5.2 PRISMATIC CONCRETE SPECIMENS REINFORCED WITH BOTH LONGITUDINAL REINFORCEMENT AND VERTICAL STIRRUPS SUBJECTED TO COMBINED BENDING AND TORSION

5.2.1 GENERAL :

In what follows hereafter, a R.C.C. specimen is meant to be a concrete specimen reinforced both longitudinally as well as with vertical stirrups. Although sufficient experimental data exists, a clear understanding of the behaviour and strength is difficult because of conflicting views put forward by various researchers. Lessig's analysis[31] appears to be promising. In this area, the least attention has been paid to the T and L-section, about which the experimental data is practically non-existent.

Tests conducted by Cowan[20,46], Lessig[30 - 33], Yudin[39], Gesund[43], Goode and Helmy[48] and Evans and Sarkar[59] give data on the effect of the following variables on the behaviour and strength of such specimens :

(i) Cross-sectional shape (Rectangular T or L or hollow Rectangular)

(ii) Compressive strength of concrete

(iii) Spacings of Stirrups

(iv) Ratio of volume of longitudinal bars to volume of vertical stirrups

(v) Ratio of applied torque to applied bending moment, i.e. $\frac{T}{M} = K$.

5.2.2 BEHAVIOUR :

The modes of failure, as hypothesised by Goode and Helmy(48) are :

1. Yielding of top bars before the stirrups or bottom bars yield - This occurs when the percentage of compression reinforcement is very small and the section is subjected to large torsional moment and small bending moment.

2. Yielding of stirrups before top or bottom bars yield: This yields the maximum torsional strength of the beam.

3. Yielding of stirrups and bottom bars simultaneously. Results show that this occurs over a considerable range of the interaction diagram for bending moment and torsion(Fig. 5.1). This mode of failure coincides with Lessig's first failure scheme (Fig. 4.4).

4. Yielding of bottom bars before the stirrups or top bars yield.

In addition to the above,

5. When the beam is "over-reinforced", the concrete crushes before the reinforcement yields. This is known as the primary compression failure and is very sudden. This can be avoided by limiting, the maximum percentage of longitudinal reinforcement or by suitable use of compression reinforcement.

LEGEND

1 : Mode 1

2 : Mode 2

3 : Mode 3

4 : Mode 4

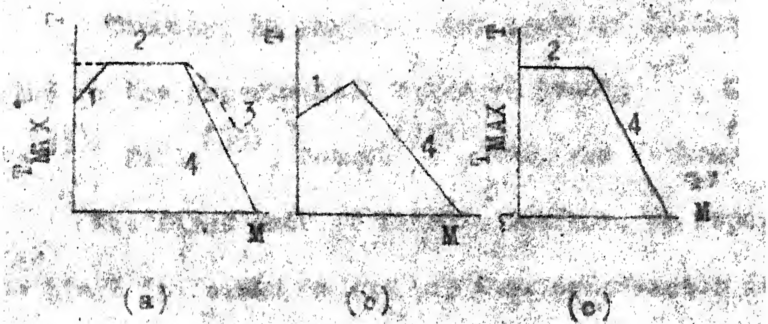


FIG. 5.1 Typical Interaction Diagrams between Torsion and Bending moment in Combined Bending and Torsion

(From Ref. 48)

6. In extremely under-reinforced beams when the torque or bending moment causing the beam to crack is larger than the ultimate strength of steel reinforcement. This can be avoided by limiting the principal tensile stress in the concrete, above which the reinforcement to be provided must be more than the nominal amount.

Figure (5.1) shows a T - M interaction diagram showing the various failure modes by numbers. These failure modes, although quite general are yet to be verified by experimental observations and quantitative analysis.

However, in general, two types of failure surfaces have been reported in the experimental works of Lessig[31], Chinenkov[32], Lialin[33], Yudin[39], Gesund[43], Evansand Sarkar[59] and Pandit[52].

The first mode of failure surface, as reported,forms when a skew hinge is formed on the top i.e. compression face (Fig. 4.4) of the specimen. This mode of failure occurs when the ratio between applied torque T, and the applied bending moment M, i.e. $K = T/M$ is < 1. Further it has been reported that the longitudinal as well as the transverse steel crossed by the failure cracks yield at the time of failure and that concrete in the compression zone (Fig. 4.4) crushes only after the yielding of all the steel in the tension zone. This means that the first-type of failure occurs when the specimen is under-reinforced with respect to the pure flexure considerations.

The second mode of failure surface has been reported to have formed when a skew hinge forms on the vertical side of the specimen (Fig. 4.5). This type of failure occurs when the ratio

K is $\geqslant$ 1 and, in general, when shear is also present. Further it has been reported that specimens over-reinforced with respect to pure flexure considerations, failed in the manner of second mode of failure.

Inter-action between T and M : Collins, Walsh[79] et al has reported the interaction curve between T and M, based on Lessig's first mode of failure, as :

$$\left(\frac{T}{T_o} \right)^2 + \left(\frac{M}{M_u} \right) = 1 \qquad \text{(Fig. 5.2)}$$

Here T_o = The pure torsional capacity of the section ; can be calculate from Lessig's second mode of failure - equati ns, when K = ∞ (pp)

and M_u = The computed ultimate flexural capacity of the section in the absence of torsion; can be obtained from Lessig's first mode of failure - equation (pp) by substituting K = 0.

Moreover, there is universal agreement among workers in the field, that combinations of longitudinal and transverse steel increases the torsional capacity ofbeams.

Iyenger and Vijayarangan[89] show an interesting case of the interaction diagram from their test results for K $\leqslant$ 0.10. They found that the bending capacity is practically not reduced because of the presence of torsi nal moment. Another interesting feature, as can be seen from (Fig: 5.3) is that the value of the torsional capacity actually increases by 15% for a bending moment of 0.65 M_u.

5.2.3 STRENGTH :

The mo st general and rational analysis up-to-date, for predicting the ultimate strength of R.C.C. rectangular specimens

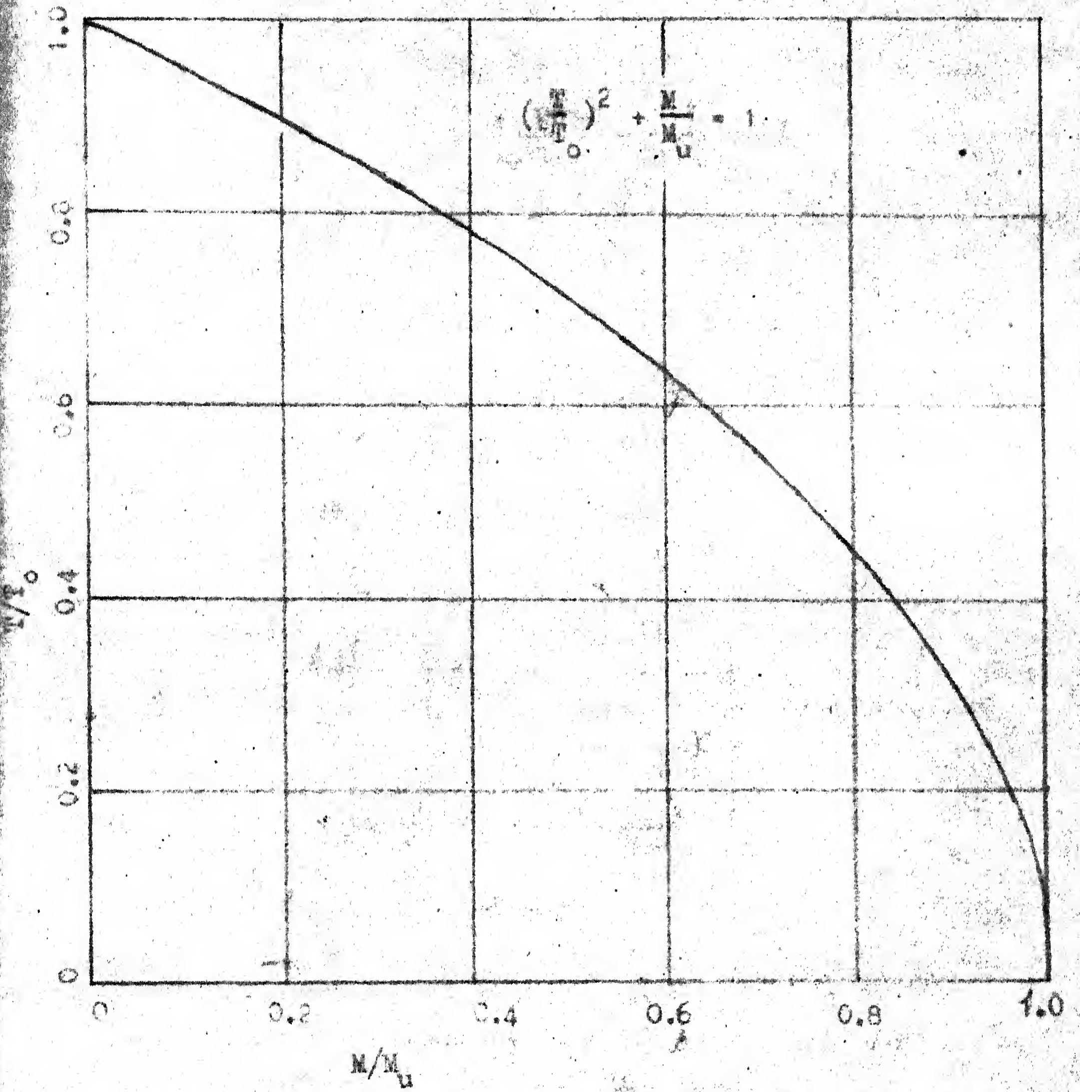
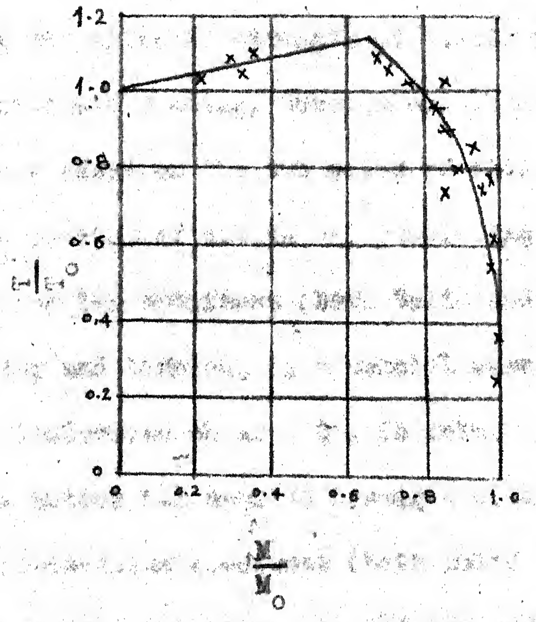


FIG. [illegible] Interaction Diagram for Beams failing in Mode 1
(From Ref. 79)



LEGEND :

T = Ultimate Torque in combined Bending and Torsion

T_o = Ultimate Torque in pure Bending

M = Ultimate bending Moment in combined Bending and Torsion

M_o = Ultimate Bending Moment in pure Bending

X = Test-results of Iyenger and Vijaya Rangan

FIG. 5.3 Interaction Diagram between Bending Moment and Torque (From Ref. 89).

subjected to combined bending, torsion and transverse shear was presented by Lessig[31] in 1958. Yudin[39] in 1962 generalised Lessig's analysis.

Lessig's theory is based on two modes of failure, determined by experiments on R.C.C. rectangular beams, subjected to any combination of loading. The first mode of failure, in which a skew hinge forms at the top (Fig. 4.4) and the second mode of failure, in which it forms on the vertical side of the specimen (Fig. 4.5). Based on these two failure mechanisms, Lessig came out with equilibrium equations meant for predicting the ultimate strength of R.C.C. rectangular specimens subjected to combined bending, torsion and shear. The details of this general analysis based on the two modes of failure, are given in Sec. 6.

The problem of design and predicting the ultimate strengths of R.C.C. rectangular specimens (both Solid and Hollow) subjected to combined bending and torsion, is a special case of Lessig's general analysis when transverse shear , Q , is zero.

The author has applied Lessig's Ultimate Equilibrium Analysis to 32 R.C.C. rectangular specimens (both solid and Hollow) subjected to combined bending and torsion, tested experimentally by Pandit[38], Evans and Sarkar[59], Gesund[43] and Walsh & Collins[89]. The comparison between the theoretical ultimate torsional strengths calculated by Lessig's analysis and the actual experimental ultimate torsional strengths reported by the researchers cited above, are given in Table Nos 5 - 8 (pp : [illegible])

In section 6 (pp: [illegible]), regarding the detailed calculations according to Lessig's analysis required in preparing these tables, are given. It is to be noted that in all these specimens tested by various

researchers, the effect of the following variables on the ultimate strength has been studied:

(i) depth-breadth ratio of the X-section

(ii) ratio of volume of longitudinal steel to volume of web steel

(iii) compressive strengths of concrete

(iv) Solid or hollow rectangular cross-sections

(v) variation of K.

From the study of tables Nos. 5 - 8 (pp:), the following inferences can be drawn :

(i) Lessig's general analysis predicts the ultimate torsional and flexural strength of both solid as well as hollow rectangular R.C.C. sections, subjected to the special case of combined bending and torsion remarkably well within an average percentage error of 9% only. Sometimes the error is on non-conservative side. For this, it appears that Yudin's generalised approach gives satisfactory results as discussed by Collins, Walsh et al[89]. However Yudin's analysis is beyond the scope of the thesis.

(ii) The specimens subjected to $K < 1$ always fail by first mode of failure (Fig. 4.4).

(iii) For design purposes, the load factor can be taken as 1.8, as suggested by Goode & Helmy[48], thus limiting the crack-width to within $\frac{1}{100}$" at working load.

(iv) The ratio of longitudinal and web reinforcements actually provided in the experiments cited above, also satisfies the upper and lower bounds of the Lessig's theory-requirements.

5.2.4 REMARKS :

In spite of the applicability of Lessig's analysis, it gives trivial results in case of concrete specimens containing longitudinal steel only, subjected to combined bending and torsion. Also, as the behaviour and strength of R.C.C., T or L sections have not so far been studied and analysed, under combined bending and torsion, it is suggested that to be on the conservative side, Lessig's analysis for R.C.C. rectangular beams may as well be applied for R.C.C., T and L sections. So in case of T or L sections, the overhanging flanges may be neglected, for want of experimental data.

SECTION 6 : PRISMATIC R.C.C. SPECIMENS SUBJECTED TO COMBINED BENDING, TORSION AND SHEAR

6.1 : PRISMATIC CONCRETE SPECIMENS, REINFORCED LONGITUDINALLY ONLY, SUBJECTED TO COMBINED BENDING, TORSION AND SHEAR.

6.1.1 GENERAL :

In whatever follows hereafter, a specimen under combined loading is meant to be a specimen under the action of combined bending, torsion and shear.

For a good understanding, the study of R.C. beams without web reinforcements under combined loading is preferable. In this area, very scanty data is available. The various researchers who have tested concrete specimens containing longitudinal steel only, subjected to combined loading, are Nylander[13] and Brown[25] who tested solid rectangular concrete specimens, Brown[25] & Kresy[62] who tested L-sections and Farmer[61] & Brown[25] who tested T-sections.

6.1.2 BEHAVIOUR :

There is a dearth of information at present regarding the behaviour of such specimens subjected to combined loading and hence no definite statement concerning the stiffness, ductility or how far the specimen behave elastically can be made

6.1.3 STRENGTH :

Hsu's Approach[56,72] :

The effect of the simultaneous application of two different types of forces on the strength of a specimen can often be expressed by an interaction curve. For example, the interaction of torsion and bending can be represented on a rectangular coordinate system.

One axis represents torsion and the other bending. The strength of a specimen subjected to a certain magnitude of torsion and bending is then described by a point on the interaction curve. The case of effect of combined loading on a concrete specimen is a three dimensional problem. Hsu(56,72) has developed a three-dimensional interaction surface to design and predict the strength of concrete specimens under combined loading.

For the combined action of bending, torsion and shear on a concrete specimen without web reinforcement, Hsu has taken three space axis, perpendicular to each other, each axis representing one type of these three forces. Then, he reports that the strength of specimen subjected to a certain magnitude of torsion, shear and bending is represented by a point on the simplified interaction surface proposed by him, based on tests conducted by Farmer(61), Ersoy(62), Brown(25) and Nylander(13).

Then torsion-bending interaction curve is actually a special case of torsion-shear-bending interaction surface. It is a curve described by the intersection of the interaction surface and a plane defined by the torsion and bending axes; similarly for other two interaction curves.

Construction of Hsu's simplified interaction surface :

Now, the interaction curves for torsion and bending moment, and for bending and shear can be obtained by tests. But the interaction curve between torsion and shear cannot be obtained by tests because it is impossible to obtain a constant shear in a finite length - of a member without the simultaneous presence of bending moment. So first of all the torque-bending and bending-shear interactions curves are obtained experimentally. Hsu interpolates the Torsion-shear interaction curves, (based on tests[13,25,61,62]) between its two bounding interaction curves

of torsion-bending and bending-shear (Fig. 6.1).

DESIGN CRITERION :

Based on the simplified intersection surface, Hsu[56,72] proposes the following design equations :

$$\left(\frac{T}{T_o}\right)^2 + \frac{Q}{Q_b} = 1 \qquad \text{for } \frac{M}{M_u} \leq 0.5 \qquad \ldots\ldots (1)$$

$$\left[\frac{T}{T_o\left(1.70 - 1.40\frac{M}{M_u}\right)}\right] + \left(\frac{Q}{Q_b}\right)^2 = 1 \qquad \ldots\ldots (2)$$

$$\text{for } 0.5 < \frac{M}{M_u} < 1.0$$

where :

T = ultimate torque capacity of the specimen under combined loading

T_o = ultimate torque capacity of the specimen under pure torsion

Q = ultimate shear - capacity of the specimen under combined loading

Q_b = ultimate shear capacity of the specimen based on diagonal cracking, under combined bending and shear.

M = ultimate bending moment capacity of the specimen, under combined loading.

M_u = ultimate resisting moment of the specimen in pure bending.

These two equations describe the inter-action surface. They apply conservatively to concrete beams of rectangular, T and L -sections reinforced with longitudinal bars only and subjected to any combination of torsion, shear and bending as verified by Hsu[56,72].

Hsu has drawn graphically the above-cited two equations, in a three dimensional model representing $\frac{T}{T_o}$, $\frac{M}{M_u}$ and $\frac{Q}{Q_b}$ along three mutually perpendicular spacial axes. The simplified interaction surface, meant for design, is shown (Fig. 6.1).

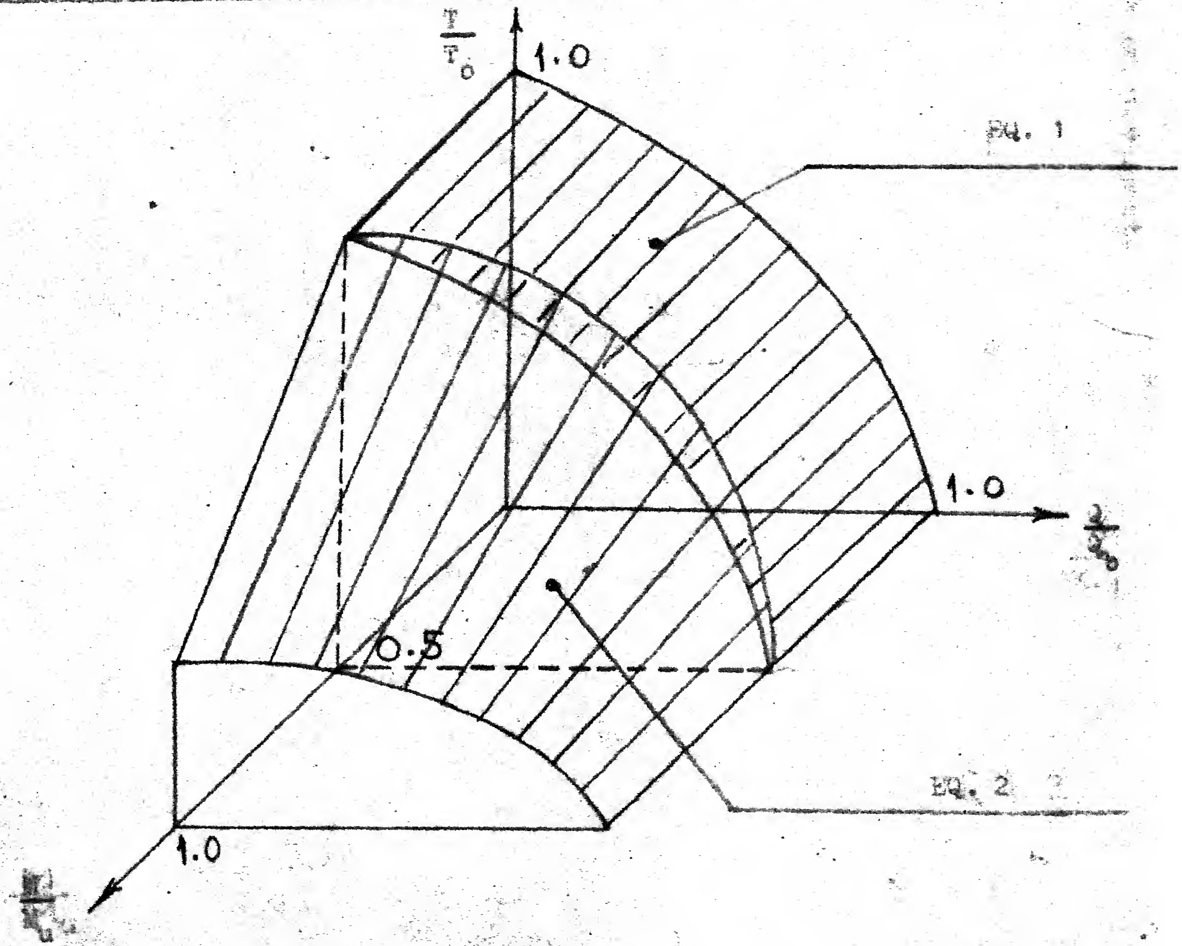


FIG. 6.1 Simplified Interaction Surface
(From Ref. 56).

6.1.4 REMARKS :

For members without web reinforcement subjected to combined loading, Hsu's results are promising and can be used for design by trial and error since Lessig's theory fails for the case of members without web reinforcement.

6.2 PRISMATIC R.C.C. SPECIMENS, WITH WEB REINFORCEMENT, SUBJECTED TO COMBINED BENDING, TORSION AND SHEAR.

6.2.1 GENERAL

The behaviour and strength of reinforced concrete members subjected to combined torsion, shear and bending, has not received enough attention in engineering literature. Although, a few test results have been recently reported and some theoretical approaches have been put forward, specially in the case of rectangular R.C.C. specimens (both solid and hollow), yet there is no experimental data available for the more common case of T and L, R.C.C. sections subjected to combined loading. It is generally agreed by the researchers that combinations of longitudinal and web steel increase the torsional capacity of beams.

6.2.2 BEHAVIOUR

A moderate study of the behaviour of R.C.C. beams subjected to combined torsion, bending and shear has been carried out in the Laboratory of Reinforced Concrete Structures - Moscow. The first major series of tests were conducted by Lessig[30,31]. Based on the tests, she observed two principal modes of failure for R.C.C. rectangular beams subjected to combined loading (Figs. 4.4, 4.5). Generally the failure of beams takes place when tension cracks on three sides open allowing the segments of the beam to rotate about a 'hinge' located near the fourth side.

In general two modes of failure have also been reported in the experimental works of Gesund[43], Lyalin[33], Chinenkov[32], Evans and Sarkar[59]. However, in 1968, Collins, Walsh et. al.[79] report that some specimens developed the skew plastic hinge near the bottom surface

and this mode labelled as mode 3, can occur for high values of K. They have given a detailed analysis for this mode also.

6.2.3 STRENGTH

Mainly two approaches have been used by various researchers to analyse and predict the ultimate strength of R.C.C. specimens subjected to combined loading. These are :

(i) Analysis based on Elastic theory or working-stress approach,

(ii) Analysis based on the Ultimate Equilibrium Method.

A brief outline of the elastic-theory approach is given below. The Ultimate Equilibrium Method shall be discussed in detail in a subsequent sub-section.

6.2.3.1 ANALYSIS BASED ON ELASTIC THEORY

This analysis is mainly due to Cowan[46]. When a R.C.C. specimen is subjected to combined loading, apart from analysing the section for pure flexure, the combined effect due to the following two types of elastic shear stresses in the section, is taken into account :

(i) Elastic shear stresses induced by bending i.e. q_s

(ii) Elastic torsional shear stresses i.e. τ

Both of these two types of shear stresses give rise to principal diagonal tensile stresses f_q and f_t respectively. These two diagonal tensile stresses are inclined at 45° to the twist-axis of the beam.

It is to be noted that there is a basic difference between diagonal tensile stress f_q and the diagonal tensile stress f_t. The former occurs in the same direction over the vertical faces only whereas the latter exists all over the periphery of the section and acts in opposite directions on the opposite vertical faces (Fig. 6.2). Consequently

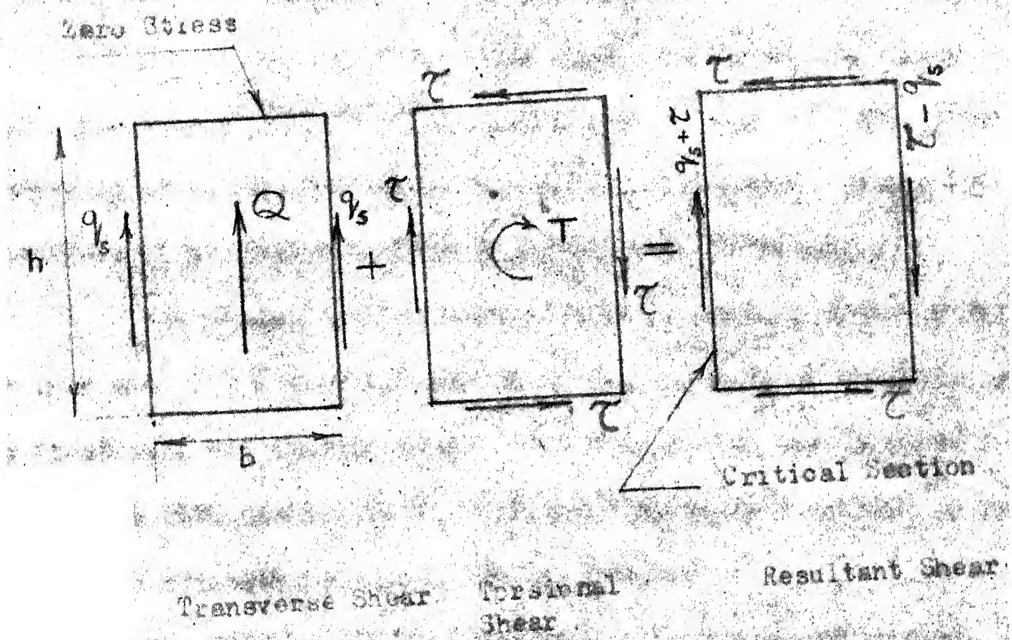


FIG. 6.2 Elastic Theory for combined Bending and Torsion

torsional and transverse shear are additive on one vertical face of the beam and subtractive on the other. The design criterion for the worst case, therefore for the combined effect of these two types of shear stresses shall be :

$$f_t + f_q \leq F_q \quad \text{where}$$

F_q is the maximum permissible concrete stress (working stress) in diagonal tension. If F_q is exceeded, the whole of the shear, torsional as well as transverse, must be taken by the reinforcement, as laid out in Indian Standard Code of Practice, I.S. 456 - 1964.

The German and Australian Codes use the elastic analysis as a basis for design(34,35,45,46). Herein, it is laid out that concrete can take part of either the transverse shear or torsional shear, but not both, the whole of remainder must be resisted by reinforcement.

The elastic stress method, however suffers from the fact that it does not give a true indication of the strength of concrete, which has an in-elastic and plastic range. But this method can be helpful in corelating with serviceability criterion which must be also satisfied along with strength criterion for the member.

The Ultimate Equilibrium Method is a strength method, rather than a stress method because it is based at failure, and the two modes of failure are generalised from experimental results. This method is discussed in the next sub-section.

6.3 ULTIMATE EQUILIBRIUM METHOD

6.3.1 INTRODUCTION

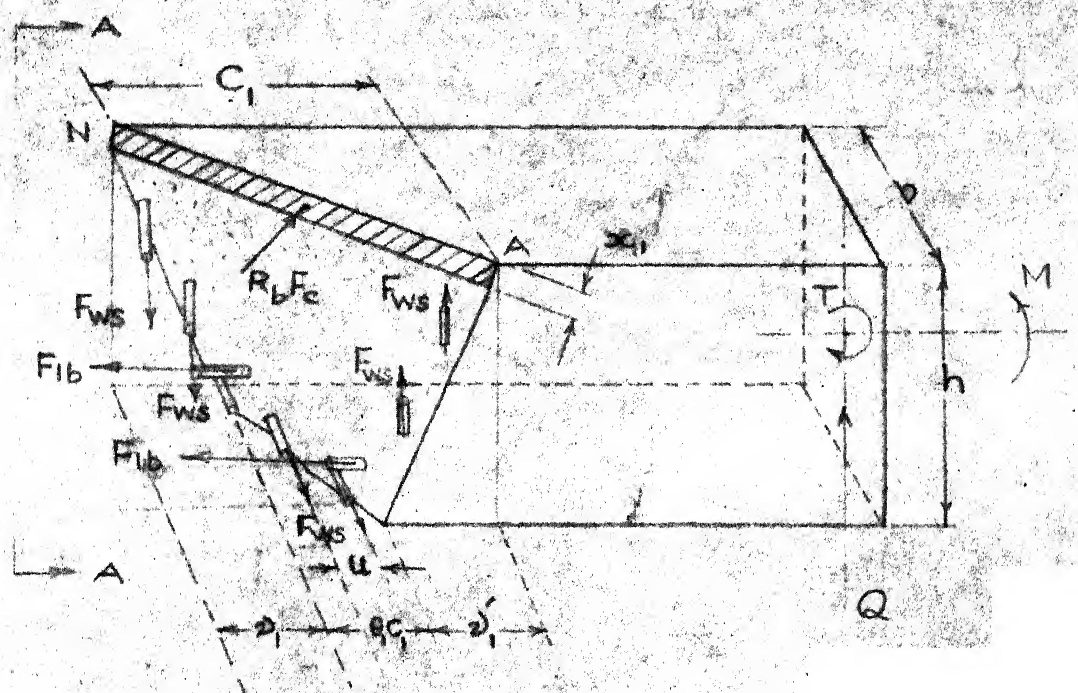
As in pure flexure, the ultimate load method is widely accepted in preference to working stress method, in the same sense,

analysis for combined loading based on the actual behaviour of the members at failing loads, gives a true insight into this complex problem. The Ultimate Equilibrium Method for the combined loading case has been put forward by Lessig[30,31], based on her test-results on such beams. She reports that R.C.C. specimens may fail in two modes already discussed in a previous sub-section.

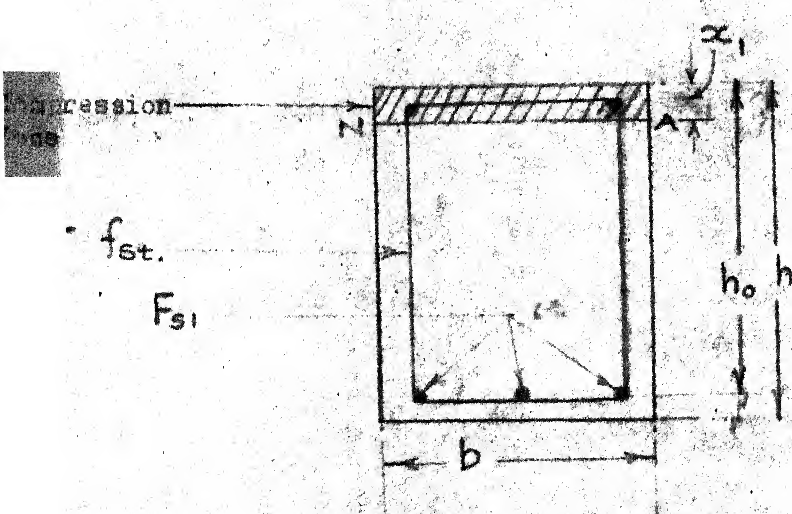
In recent years, several investigators have proposed theories to calculate the ultimate strength of beams subjected to combined loading. It is generally agreed that failure of the beams takes place when the two segments of the beam rotate about a hinge located on one side and opening of cracks on the other three sides (Figs. 4.4 and 4.5).

The first mode of failure (Fig. 4.4), as reported by Lessig, occurs most frequently, with beams subjected to combined bending and torsion with negligible transverse shear. Cracks form on the sides and in the soffit of the beam. The opening of these cracks starts only when both longitudinal and web reinforcement in the tension zone, yield, and a skew hinge forms at the top face. Then the two segments of the beam rotate about this skew hinge (called neutral axis - NA), until the concrete in compression zone crushes. So it is a secondary compression failure case and most likely, this failure occurs when the beam is under-reinforced with respect to pure flexural considerations.

This mode of failure is most likely to occur in case of beams subjected to heavy bending and small torsional moment, i.e. when $\frac{T}{M} = K < 1$, as is evident from Table Nos. (1 - 9) and also shown experimentally by Walsh, Collins et. al.[79].



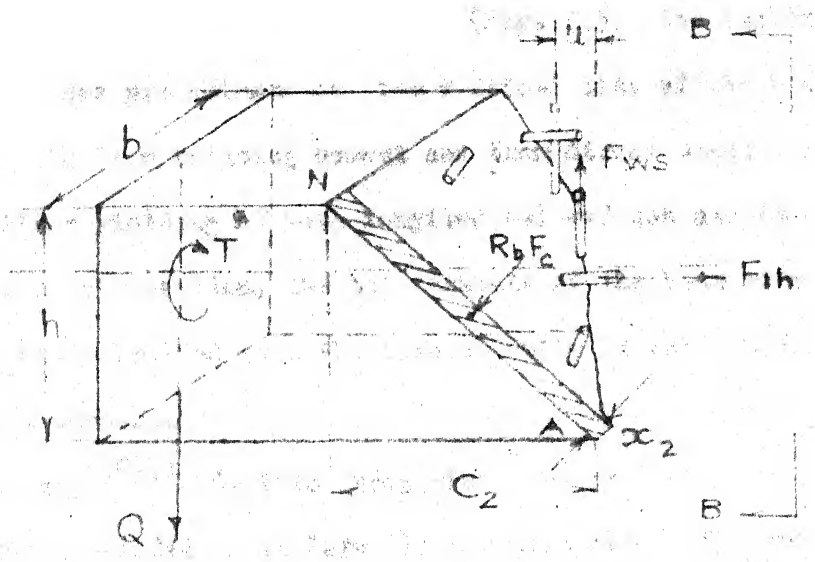
LONGITUDINAL ELEVATION



SIDE ELEVATION - AA

LESSIG'S FAILURE MODE 1.

FIG. 4.4 (From Ref. 79).


B
b
N
F_{ws}
M
T
R_bF_c
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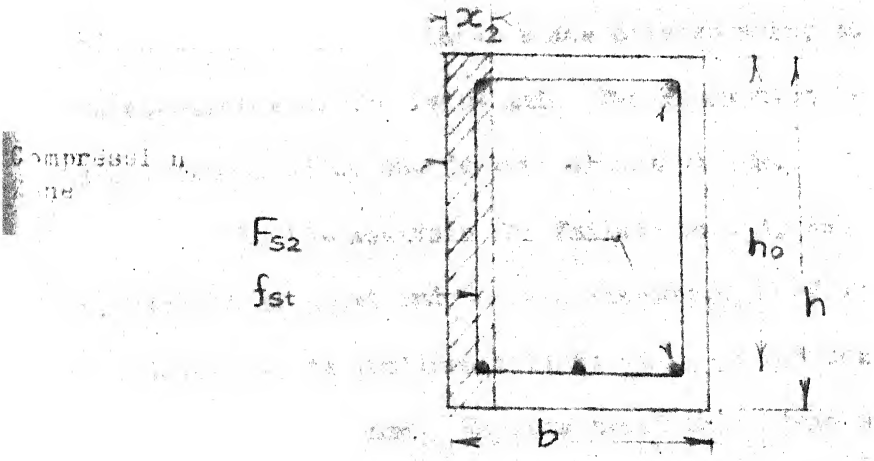
h
x_2
C_2
B
Q


x_2
Compression
zone
F_{s2}
f_{st}
h_o
h
b

The second mode of failure occurs, in beams subjected to combined loading, in which shear force is substantial and also in cases where $K \geq 1$ or in beams over reinforced with respect to pure flexural considerations. In this mode of failure (Fig. 4.5), the inclined tension cracks are predominant on the vertical side of the beam, where tensions arising from twisting moment and from direct shear force are additive. After yielding of both longitudinal and web steels in the tension zone of the section, the two segments of the beam rotate about an inclined hinge located near the face opposite to that in which tension cracks first appears.

Lessig[30,31] derives expressions for predicting the ultimate loads for the two modes of failure already described. For the purpose of analysis, she assumes that the intersections of the failure surface with the beams faces are straight lines and further that the inclination of these lines on the three sides corresponding to the tension cracks is constant with the twist axis. The assumption is also made that all steel traversed by the failure cracks yields.

In the analysis for failure mode 1, the stresses in the compressional steel and the tensile strength of uncracked concrete are ignored. An inclined failure hinge of undetermined length ℓ, (Fig. 4.4) is assumed. Moments about this hinge are calculated for the tensile forces in the longitudinal reinforcement, and the horizontal and vertical legs of the stirrups. These moments are then equated to the components of the external moments about this axis. The critical length of the failure hinge which makes the moment a minimum is then determined. The depth of the compression zone is then found by equating

the compressive forces perpendicular to the failure surface to the components of the tensile forces in the steel. The depth of the compression zone computed on this basis is quite small.

6.3.2 DERIVATION OF LESSIG'S FORMULAE

Lessig[30,31] observed two types of failure mechanisms, already described in a previous sub-section. Based on the two failure modes, Lessig[30,31] derived the ultimate-strength predicting formulae for R.C.C. rectangular beams subjected to combined loading, assuming the following :

(i) The longitudinal and web reinforcements crossed by a special tension crack, yield at failure. This assumption has been found to be practically valid, in the experimental works of Gesund[43] et. al., Evans and Sarkar[59], Pandit[52], Hsu[57], Lyalin[33] and Yudin[39,44].

(ii) The compression force in the longitudinal reinforcement located in the compression concrete zone (shown hatched in Figs : 4.4, 4.5), is negligible and can be ignored. This assumption is also practically feasible, as found by the above-cited researchers.

(iii) The stirrups are uniformly distributed throughout the span of the beam. This can be achieved easily by keeping uniform spacing of stirrups, which must be done in beams subjected to combined loading.

(iv) The compression concrete zone is rectangular. This has been proved to be true by Lessig, through theoretical calculations.

(v) The tensile strength of concrete is neglected. This is desirable because this assumption gives some conservative results which is required in the complex problem of R.C.C. beams subjected to combined loading.

Lessig derived her strength-formulae, on the following two equilibrium equations :

(i) The algebraic sum of the moments of all external and internal forces, about the neutral axis NA (plastic skew hinge; Figs : 4.4, 4.5) is zero.

(ii) The algebraic sum of all the force-components, external aswell as internal in the direction of the normal to the concrete compression zone is zero.

Lessig's formulae for predicting the ultimate strength based on the above-cited assumptions and employing the two equilibrium equations, are given in the next sub-section.

Chinenkov(32), who conducted a series of tests on reinforced concrete beams subjected to combined loading concluded that Lessig's analysis agrees substantially well with test values, although on the conservative side. Experimental values slightly higher than the predicted values could be accounted for by the concrete tension strength ignored in Lessig's analysis.

Yudin(39,44) observed that the analysis employed by Lessig, only satisfied two equations of equilibrium. This criticism was also mentioned by Hsu(89) while discussing Gesund's work(43). To satisfy equilibrium completely, it is necessary to consider shear and other

forces in the compression zone. Yudin, however, ignored the distribution of stresses in the compression zone and his analysis also fails to completely satisfy equilibrium. Gesund[43] et. al. developed an Ultimate Equilibrium theory for combined bending and torsion using an approach very similar to that of Yudin. They consider the moments about longitudinal and transverse axis. Collins, Walsh[89] et. al. report in a discussion to Gesund's work that Lessig's analysis for $K > 0.5$ givesvalues on the unsafe side although the average error is 7% ; Yudin's analysis is conservative and moderately accurate (the average error for about 22 beams is 15%) and Gesund's analysis is more conservative than Yudin (average error of 21%). A detailed check of the validity of Lessig's analysis is compared in Table Nos. (1 - 9) in a subsequent sub-section. It is reported recently that for design purposes Yudin's formulae are very much simpler than those of Lessig. However Yudin's work is outside the scope of the thesis.

As the problem of failure under combined loading is a 3-dimensional one, six equilibrium equations need be satisfied. But according to Lessig, only the two above-cited equations need be satisfied, as she arguesthat after the formation of the plastic skew hinge NA, rotation is possible only about it. This seems to be a draw-back of Lessig's theory and Yudin generalised Lessig's theory by employing three equilibrium equations.

In deriving Lessig's formulae, the following notations shall be used :

T, M, Q = respectively the external twisting moment, bending moment and transverse shear force at a cross-section at failure.

h, b = overall depth and breadth of the section.

h_o = effective depth of the section.

a = uniform concrete-cover alround the four sides of the section

u = spacing of the closed stirrups.

NA = neutral axis or plastic skew hinge.

C_1, C_2 = projections of the neutral axis in failure modes 1 and 2 respectively.

F_{s1} = total cross-sectional area of all longitudinal reinforcement near the face of width b, in tension (mode 1 : Fig. 4.4)

F_{s2} = total cross-sectional area of all longitudinal reinforcement near the face of height h, in tension (mode 2 : Fig.4.5).

f_{st} = cross-sectional area of the material from which stirrups made.

x_1, x_2 = respectively the depth of compression stress block for failure mode 1 and mode 2.

x_{na} = depth of the neutral axis $= \frac{x}{K_1}$.

K_1 = ratio of the area of the concrete stress block at ultimate to the rectangular block.

σ_{st} = yield stress in web-steel.

σ_s = yield stress in longitudinal steel.

R_b = crushing concrete strength in compression due to bending

$= 0.85 f'_c$.

f'_c = compressive cylinder strength of concrete.

K $= T/M$

λ $= 2T/Qb$

= inclined length of the plastic skew hinge NA.

F_{1b} $= \sigma_s . F_{s1}$

F_{WS} $= \sigma_{st} . f_{st}$.

F_c = area of rectangular compression concrete zone

F_{1h} = $_s F_{s2}$

β = constant crack angle with the horizontal axis.

6.3.2.1 DERIVATION OF LESSIG'S ANALYSIS FOR R.C.C. BEAMS SUBJECTED TO COMBINED LOADING BASED ON FAILURE MODE 1 (FIG. 4.4) :

Calculation of moment due to external forces i.e. $M_{Ext.}$ about NA :

$$M_{Ext.} = M \frac{b}{\ell} + T \frac{C_1}{\ell} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Q, the transverse shear force does not appear in the expression above, as its moment-arm is zero.

Calculations of moments of internal forces, i.e., $M_{Int.}$, about NA :

Mint. comprises of the following component internal moments :

(i) moment of tension force in longitudinal steel i.e. $M_{\ell s}$ about NA:

$$M_{\ell s} = \sigma_s F_{s1} (h_o - x_1) \frac{b}{\ell} \quad \ldots\ldots\ldots\ldots \quad (2)$$

(ii) moment of tension forces in the horizontal branches of stirrups i.e. $M_{sh.}$ about NA :

$$M_{sh} = \frac{f_{st.}}{u} \sigma_{st} \theta_1 C_1 (h - a - x) \frac{C_1}{\ell} \quad \ldots\ldots \quad (3)$$

(iii) moment of the compression force in compressional concrete zone i.e. M_c about NA :

$$M_c = R_b . \frac{(C^2_1 + b^2)^{\frac{1}{2}}}{3} x^2_1 \quad \ldots\ldots\ldots\ldots \quad (4)$$

(iv) The forces in the two vertical branches of the stirrups is denoted by P, in each branch (Fig. 6.3).

$$\text{Now } P = \sigma_{st.} \frac{f_{st}}{u} \left(\nu_1 - a \frac{C_1}{b} \right)$$

It is known from theoretical mechanics that the moment of forces P about NAis equal to projection on this axis of the moment of force P with respect to point O, taken on NA, arbitrarily :

moment of P about O, M_{PO} = (moment of P about NA, $M_{P\ell}$) X Sec. ϕ

where ϕ is the angle between the moment vectors M_{PO} and $M_{P\ell}$ (Fig.6.3).

$$\text{Now } M_{PO} = P\left(\nu_1 - a \frac{C_1}{b}\right)$$

$$\text{and } 2\nu_1 = (1 - \theta_1)\, C_1$$

If β is the constant crack angle with respect to a horizontal axis, then

$$\text{Cot } \beta = \frac{2\nu_1}{2h - 2x_1}$$

Now $M_{P\ell} = M_{PO}$ Cos ϕ where Cos $\phi = \frac{b}{\ell}$.

So, finally the following expression is obtained after substituting various values :

$$M_{P\ell} = \sigma_{st} \frac{f_{st} C_1^2}{2\, u\, b\, \ell} \left\{ \left[(1 - \theta_1)\, b \frac{h - x_1}{2h - 2x_1} - a \right]^2 + \right.$$

$$\left. + \left[(1 - \theta_1)\, b \frac{h - x_1}{2h - 2x_1} - a \right]^2 \right\} \quad \ldots\ldots\ldots \quad (5)$$

$$\text{Now } M_{Int.} = M_{\ell s} + M_{sh} + M_{e} + M_{P\ell}$$

$$= M_{Ext.}$$

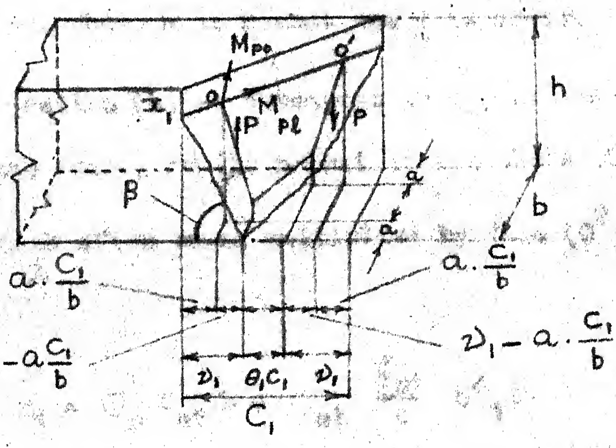


FIG. 6.3 Determination of the Moment with respect to the axis ℓ, due to the stresses in the vertical branches of the Stirrups (From Ref. 46).

So the following expression is obtained :

$$Mb + T\,C_1 = R_b(C^2_1 + b^2)\,\frac{x^2_1}{2} + \sigma_s F_{s1}\, b\,(h_o - x_1) +$$

$$+ \sigma_{st}\frac{f_{st}}{u}\,C^2_1\,\theta_1\,(h - a - x_1) +$$

$$+ \sigma_{st}\frac{f_{st}\,C^2_1}{u\,b}\left[(1 - \theta_1)\,\frac{b}{2} - a\right]^2 \quad .... \quad (6)$$

Now, the second equilibrium equation is applied :

It is known that $\frac{dM}{dX} = Q$ where M is moment and Q is shear.

So differentiating equation (6) with respect to x_1, an equation for the projections of all the forces on the normal to the plane of compression zone is obtained, which after being multiplied by $\ell = (C^2_1 + b^2)^{\frac{1}{2}}$, reduces to :

$$R_b\,(C^2_1 + b^2)\,x_1 = \sigma_s F_{s1}\,b + \sigma_{st}\,\frac{f_{st}}{u}\,C^2_1\,\theta_1 \quad ... \quad (7)$$

Multiplying equation (7) by $\frac{1}{2}\,x_1$ and subtracting the equation thus obtained from equation (6), the following equation results :

$$Mb + T\,C_1 = \sigma_s F_{s1}\,b\,(h_o - \frac{x_1}{2}) + \sigma_{st}\frac{f_{st}}{u}\,C^2_1\,\theta_1\,(h - a - \frac{x_1}{2}) +$$

$$+ \sigma_{st}\frac{f_{st}}{u}\,C^2_1\,\frac{b}{4}\,(1 - \theta_1 - \frac{2a}{b})^2 \quad .... \quad (8)$$

Now according to the assumptions, that crack-angle, β is constant, if the beam sides are unfolded, the crack would thus form practically a straight line (Fig. 6.4). The angle β is never less than 45° for combined loading case.

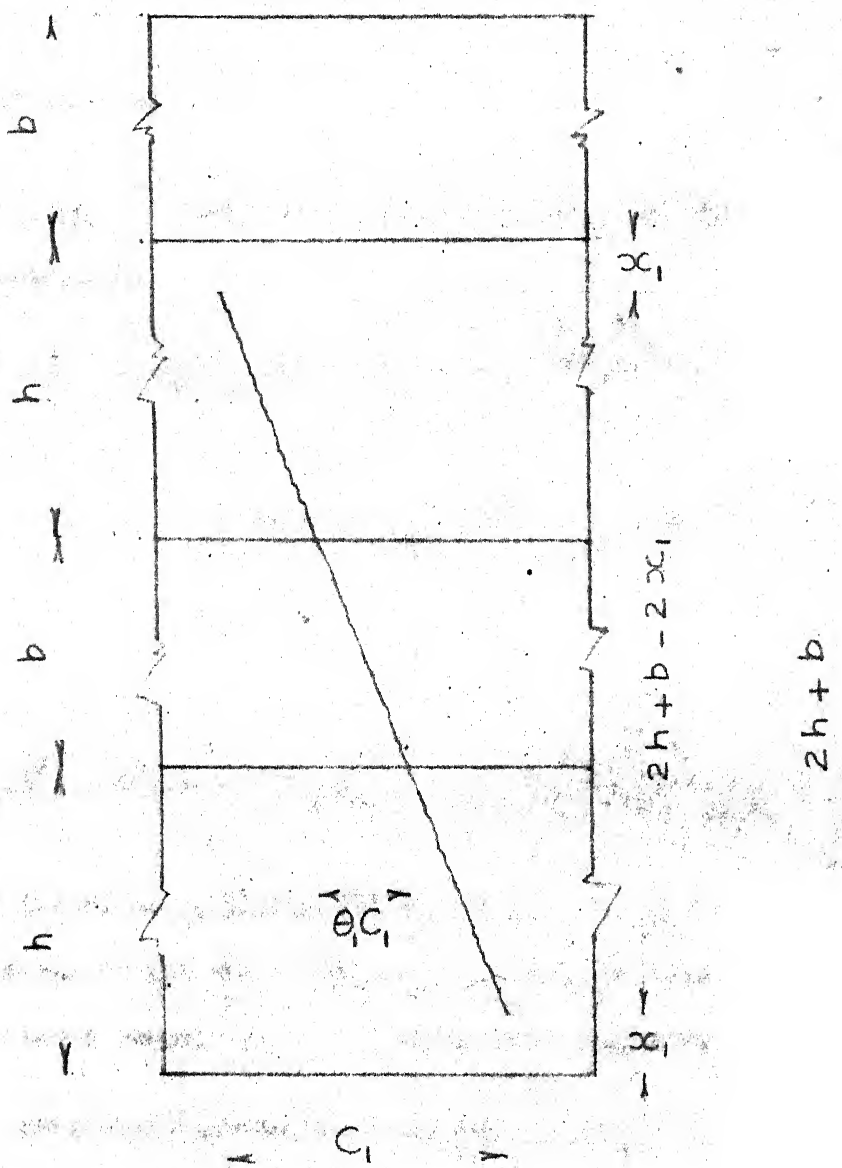


FIG. 6.4 Beam-sides Unfolded
(From Ref.46)

so $\quad \theta_1 = \dfrac{b}{2h + b - 2x_1}$ . . . . (9)

and denoting , $K = \dfrac{T}{M}$ . . . . (10)

Substituting θ_1 and K in equations (7) and (8), the following equations result :

$$T\left[\frac{C_1}{b} + \frac{1}{K}\right] = \sigma_s F_{s1}\left(h_o - \frac{x_1}{2}\right) + \sigma_{st}\frac{f_{st}C^2_1}{u} \cdot \frac{h - a - \frac{1}{2}x_1}{2h + b - 2x_1} +$$

$$+ \sigma_{st}\frac{f_{st}C^2_1}{u}\left(\frac{h - x_1}{2h + b - 2x_1} - \frac{a}{b}\right)^2 \quad \ldots\ldots \quad (11)$$

and

$$R_b(C^2_1 + b^2)\,x_1 = \left[\sigma_s F_{s1} + \sigma_{st}\frac{f_{st}C^2_1}{u\,(2h + b - 2x_1)}\right] b \quad \ldots \quad (12)$$

To get the value of the projection of NA, i.e. C_1, corresponding to the minimum resistance of the beam section, differentiate equation (11) without respect to C_1 and equate the result to zero i.e. $\frac{\partial T}{\partial C_1} = 0$;

So the following expression is obtained for C_1 :

$$C_1 = -\frac{b}{K} + \left[\left(\frac{b}{K}\right)^2 + \frac{h_o - \frac{1}{2}x_1}{\frac{\sigma_{st}}{\sigma_s}\frac{f_{st}}{F_{s1}} \cdot \frac{1}{u} \cdot \frac{h-a-\frac{1}{2}x_1}{2h+b-2x_1} + \frac{\sigma_{st}}{\sigma_s}\frac{f_{st}}{F_{s1}} \cdot \frac{1}{u}\left(\frac{h - x_1}{2h+b - 2x_1} - \frac{a}{b}\right)^2}\right]^{\frac{1}{2}}$$

$$\ldots\ldots \quad (13)$$

From experiments(30,31,33,44) conducted in Russia, it has been found that the value of $\theta_1 = \dfrac{b}{2h + b - 2x_1}$ used in the above analysis, is always a little less than that used in the analysis. So, θ_1 can be

taken as : $\theta_1 = \dfrac{b}{2h + b}$ ($2x_1$ being negligible as compared with $2h + b$).

Further simplification can be done by taking $h - a - x_1/2 \approx h_o - x_1/2$.

Also in equation (11), the last member in the right hand side and the corresponding value under the square root in equation (13) are negligibly small and can be neglected.

So after incorporating, the simplifications, cited above, the following final strength-formulae, based on the mechanism of failure mode 1, are obtained :

$$T\left(\frac{C_1}{b} + \frac{1}{K}\right) = \left[\sigma_s F_{s1} + \sigma_{st} \frac{f_{st} C_1^2}{u(2h + b)}\right] \cdot \left(h_o - \frac{x_1}{2}\right) \qquad (14)$$

$$R_b(C_1^2 + b^2)\, x_1 = \left[\sigma_s F_{s1} + \sigma_{st} \frac{f_{st} C_1^2}{u(2h + b)}\right] b \quad \dots\dots \quad (15)$$

$$C_1 = -\frac{b}{K} + \left[\left(\frac{b}{K}\right)^2 + \frac{\sigma_s F_{s1} u}{\sigma_{st} f_{st}} (2h + b)\right]^{\frac{1}{2}} \quad \dots\dots \quad (16)$$

$$\text{and} \quad C_1 \leq 2h + b \quad \dots\dots\dots\dots\dots \quad (17)$$

The last inequality is based on the fact that the crack-angle β is never less than 45° in combined loading case (Fig. 6.4).

Equations (14,15) can be checked with the degenerate case when no torsion is present i.e. $C_1 = 0$ and for so called "under reinforced" sections in flexure. Eqn. (15) gives the value of the depth of the stress block and Eqn. (14) givesthe ultimate flexural capacity. Extracts of flexural theory for R.C.C. sections is given in Section 6.4.1.

6.3.2.2. DERIVATION OF LESSIG'S ANALYSIS FOR R.C.C. BEAMS SUBJECTED TO COMBINED LOADING BASED ON FAILURE MODE 2 (FIG. 4.5) :

The strength predicting formulae for this mode can be derived exactly in the same manner and by similar steps, as for failure mode 1, incorporating parallel simplifications, and by interchanging the places of b and h in the formulae for failure mode 1.

The following final strength formulae, according failure mode 2 are obtained :

$$T.\frac{C_2}{h}\left(1+\frac{1}{\lambda}\right) = \left[\sigma_s F_{s2} + \sigma_{st}\frac{f_{st}\, C_2^{\,2}}{u(2b+h)}\right]\left(b - a - \frac{x_2}{2}\right) \qquad (18)$$

$$R_b(C_2^{\,2} + h^2)\, x_2 = \left[\sigma_s F_{s2} + \sigma_{st}\cdot\frac{f_{st}\, C_2^{\,2}}{u(2b+h)}\right] h \quad \ldots\ldots \qquad (19)$$

$$C_2 = \left[\left(\frac{\sigma_s F_{s2}\, u}{\sigma_{st} f_{st}}\right) . (2b+h)\right]^{\frac{1}{2}} \quad \ldots\ldots\ldots \qquad (20)$$

$$\text{and } C_2 \le 2b + h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (21)$$

Then the theoretical minimum torque capacity of R.C.C. rectangular beams subjected to any combination of loading is given by either equations (14) or (18), whichever gives lesser capacity of the section.

It is to be noted that formulae for failure modes 1 and 2 derived above, are based on the assumption that both longitudinal and web steels in the tension zone yield. This condition imposes several limitations on the amount and distribution of reinforcement, which may be used.

In the first instant, the amount of reinforcement must be limited, so that the concrete in the compression zone will not crush

before the reinforcement yields. So the maximum depth of the compression zone, x_1, has to be established. Based on experiments conducted in Russia[30,31,46], it has been found that x_1 must satisfy the following relation :

$$x_1 \leq \left[0.55 - 0.7\,(K)^{\frac{1}{2}}\right] \cdot h_o \text{ for } 0 < K < 0.2 \qquad (22)$$

For maximum value of x_2, there has been no report from any researcher and further study is needed.

Lessig[30,31] and Lyalin[33] attempted to establish empirically the limits of the ratio of transverse to longitudinal steel for which yielding of both steels could be guaranteed. They fixed the limits, based on experiments, as follows :

From failure mode 1 :

$$0.5 \leq \frac{\sigma_{st}\, f_{st}\, b}{\sigma_s\, F_{s1}'\, u} \left[1 + \frac{2}{K}\left(\frac{b}{2h + b}\right)^{\frac{1}{2}}\right] \leq 1.5 \qquad (23)$$

For failure mode 2 :

$$0.5 \leq \frac{\sigma_{st}\, f_{st}\, h}{\sigma_s\, F_{s2}\, u} \leq 1.5 \quad \dots\dots\dots\dots \qquad (24)$$

From plastic theory for torsion due to Nadai[15] for plain concrete rectangular specimens, no web steel is required if :

$$T < \frac{1}{6} R_t\, b^2\, (3h - b) \quad \dots\dots\dots \qquad (25)$$

and $K \leq 0.2$

where R_t = tensile strength of concrete at ultimate loading.

The Ultimate Equilibrium Method takes into consideration, the crack-control criterion, and based on experiments[46], the following condition must be satisfied, by the chosen dimensions of the cross-section of the beam :

$$T \leq 0.07\, R_b\, b^2\, h \qquad \ldots\ldots\ldots\ldots \qquad (26)$$

The limits of ratio as cited above between areas of longitudinal and web steels, have been empirically fixed by Lessig and Lyalin based on their rather small number of experiments. According to them these limits ensure the section to be under reinforced , so that reinforcement yields prior to crushing of concrete at failure. The failure, by observing these limits, is delayed and gives enough warning, which is desirable in designing sections according to such delayed failures. Failure Mode 1, takes place in sections subjected to combined bending and torsion, in which bending is predominant and this failure is not sudden, and design of sections is recommended to be done, based on this mode of failure. Failure Mode 2, is not well established analytically and, being a shear failure is sudden and destructive and so, it should be avoided while designing R.C.C.sections.

From the observations of Table Nos. (1 - 9), it is clear that the upper limit fixed on the ratio between the areas of longitudinal and web reinforcements, for the two modes of failure, can be relaxed, and further experimentation is desired to arrive at a conclusive upper limit.

6.3.2.3 LIMITATIONS OF LESSIG'S APPROACH :

(i) Lyalin reports that the analysis gives good agreement with experimental data only in cases where at the time of failure, both longitudinal and web reinforcements have yielded.

(ii) The restrictions imposed on the ratio between longitudinal and web steel areas, are required to be modified and relaxed, and so more experiments are needed.

(iii) The failure mode 2 is yet to be verified with more experimental failures in this mode.

(iv) Lessig's analysis is limited to $K \leq 0.2$.

(v) The empirical equation (22) is based on rather small number of tests conducted by Lessig, and she suggested that there is need for further work in this aspect of the problem.

(vi) Lessig's method is an analysis approach and design must be carried by assuming all dimensions and reinforcements then checking the capacity, and if necessary modifying the section and repeating the process.

(vii) The analysis fails when the specimens are reinforced longitudinally only.

(viii) In case of specimens over reinforced or in beams in which torsional and transverse shear loading is predominant, and in which a sudden failure is expected, Lessig's analysis requires to be further elaborated, modified, and put to test with more experimentation in this direction.

6.3.3 VALIDITY OF LESSIG'S ULTIMATE EQUILIBRIUM METHOD :

Various researchers have tested R.C.C. rectangular sections (solid as well as hollow) under pure torsion, torsion combined with bending, and combined bending, torsion and shear. The author has calculated the theoretical ultimate torque values of about 100 R.C.C. rectangular specimens subjected to various combinations of loading, experimentally tested by Pandit[38], Gesund[43] et. al. , Hsu[55], Evans and Sarkar[59], Lyalin[33] and Collins et. al.[89]. The minimum theoretically calculated torque capacity according to Lessig's strength formulae [Equations (14) or (18) which-ever gives lesser value of T] is compared with the actual experimental value reported by the above-cited researchers. The comparison of $\frac{T_{cal}}{T_{exp}}$ is given in the Table Nos.(1-9). It is to be noted that all about 100 specimens checked in these tables are both longitudinally and transversely reinforced.

6.3.3.1 EXPLANATION OF THE TABLES :

The second column in (Table Nos. 1-7, 9) is (b x h) and gives the values of width and overall depth of the specimens respectively.

The third column in Table No. (9), fourth column in Table Nos. (1 - 8) gives the experimental value for $K = \frac{T}{M}$, a ratio between the failing torque and the failing flexure moment, in combined loading, as reported in the experiments. The fifth column in Table Nos. (1-7, 9) gives the mode of failure (first or second) by which the least torque capacity of the specimen is obtained theoretically by Lessig's strength formulae [Equations (14) or (18)] . The third column in Table Nos. (1 - 7) gives the ratio between the transverse and longitudinal steel

SOLID RECTANGULAR R.C.C. BEAMS UNDER PURE TORSION

SOURCE : G.S.PANDIT[38]

TABLE 1

(1) Beam No.	(2) b x h (in. x in.)	(3) Equation (24)	(4) $K = \frac{T}{M}$	(5) Failure mode - applicable	(6) $\frac{T_{cal.}}{T_{exp.}}$	Error in Calculated Torque (7) %age error	(8) Mean percentage error
B-4	6 x 12	0.67		II	1.26	26	
C-4	6 x 12	0.67		II	1.26	26	17.33
E-3	6 x 12	0.67		II	1.00	0	

SOURCE : T.C.HSU[55]

TABLE 2

(1) Beam No.	(2) b x h (in. x in.)	(3) Equation (24)	(4) $K = \frac{T}{M}$	(5) Failure mode - applicable	(6) $\frac{T_{cal.}}{T_{exp.}}$	(7) %age error	(8) Mean percentage error
B-1	10 x 15	0.62		II	0.84	16	
B-2	10 x 15	0.62		II	0.98	2	
B-3	10 x 15	0.62		II	1.00	0	
B-4	10 x 15	0.62		II	1.12	12	
B-5	10 x 15	0.62		II	1.28	28	
B-6	10 x 15	0.62		II	1.50	50	
B-7	10 x 15	1.35		II	1.00	0	

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
B-8	10 x 15	3.00		II	1.23	23	
M_1	10 x 15	0.42		II	0.79	21	
M_2	10 x 15	0.42		II	0.80	20	
M_3	10 x 15	0.42		II	1.00	0	
M_4	10 x 15	0.42		II	1.12	12	
M_5	10 x 15	0.42		II	1.27	27	
I-2	10 x 15	0.62		II	0.76	24	
I-3	10 x 15	0.62		II	0.85	15	
I-4	10 x 15	0.62		II	0.92	8	
I-5	10 x 15	0.62		II	0.98	2	
I-6	10 x 15	0.62		II	1.05	5	
J-1	10 x 15	0.62		II	0.82	18	
J-2	10 x 15	0.62		II	0.93	7	
J-3	10 x 15	0.62		II	1.08	8	
J-4	10 x 15	0.62		II	1.30	30	
G-1	10 x 20	0.70		II	0.76	24	
G-2	10 x 20	0.70		II	0.80	20	

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
G-3	10 x 20	0.70		II	0.92	8	
G-4	10 x 20	0.70		II	0.95	5	
G-5	10 x 20	0.70		II	1.05	5	
N-1	6 x 12	0.70		II	0.72	28	
N-2	6 x 12	0.70		II	0.83	17	
C-1	10 x 10	0.53		II	0.73	27	
C-2	10 x 10	0.53		II	1.00	0	17%
C-3	10 x 10	0.53		II	1.10	10	
C-4	10 x 10	0.53		II	1.30	30	
C-5	10 x 10	0.53		II	1.50	50	
C-6	10 x 10	0.58		II	1.70	70	

SOURCE : I.C.

(1) Beam No.	(2) b x h (in. x in.)	(3) Equation (24)	(4) K	(5) Failure Scheme Applicable	(6) $\frac{T_{cal.}}{T_{expt.}}$	Error in Calculated Torque (7) %age error	(8) Mean %age error
D-1	10 x 15	0.62		II	0.75	25	
D-2	10 x 15	0.62		II	0.90	10	11
D-3	10 x 15	0.62		II	0.94	6	
D-4	10 x 15	0.62		II	1.04	4	

SOURCE : EVANS AND SARKAR[59]

TABLE - 4

(1) Beam No.	(2) b x h (in. x in.)	(3) Equation (24)	(4) K	(5) Failure Scheme Applicable	(6) $\frac{T_{cal.}}{T_{expt.}}$	(7) %age error	(8) Mean %age error
HB/1	$6\frac{1}{16} \times 7\frac{5}{8}$	0.34		I	0.76	24	
HB/7	6 x 9	0.45		II	1.10	10	14
HB/13	6 x 12	0.83		II	0.72	28	

SOLID RECTANGULAR R.C.C. BEAMS, UNDER COMBINED BENDING AND TORSION

SOURCE : GESUND ET AL[43]

TABLE - 5

(1) Beam No.	(2) b x h (in. x in.)	(3) Equation (23)	(4) K	(5) Failure mode Applicable	(6) $\frac{T_{cal.}}{T_{expt.}}$	Error in Calculated Torque (7) %age error	(8) Mean %age error
1	8 x 8	0.62	1.00	I	1.05	5	
2	8 x 8	1.56	1.00	I	1.05	5	
3	8 x 8	0.95	0.50	I	1.02	2	
4	8 x 8	2.37	0.50	I	1.08	8	
5	8 x 8	1.26	0.33	I	0.90	10	
6	8 x 8	3.17	0.33	I	1.12	12	
7	8 x 8	1.60	0.25	I	0.98	2	8.0
8	8 x 8	4.00	0.25	I	0.98	2	
9	6 x 12	0.50	0.50	I	1.12	12	
10	6 x 12	0.60	0.25	I	1.12	12	
111	6 x 12	0.75	0.50	I	1.22	22	
12	6 x 12	1.18	0.25	I	1.05	5	

R.C.C. SOLID RECTANGULAR R.C.C. BEAMS SUBJECTED TO COMBINED BENDING AND TORSION

SOURCE : PANDIT[38]

TABLE - 6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
B-2	6 x 12	0.93	0.37	I	0.92	8	
B-3	6 x 12	0.58	0.86	I	1.26	26	
C-I	6 x 12	1.00	0.28	I	1.00	0	
C-II	6 x 12	0.62	0.54	I	0.87	13	9
C-III	6 x 12	0.46	1.00	I	0.87	13	
D-I	6 x 12	0.68	0.16	I	1.05	5	
E-I	6 x 12	1.25	0.92	I	0.94	6	
E-II	6 x 12	0.76	0.92	I	1.00	0	

HOLLOW R.C.C. RECTANGULAR BEAMS SUBJECTED TO COMBINED BENDING AND TORSION

SOURCE : EVANS AND SARKAR(59)

TABLE - 7

(1) Beam No.	(2) b x h (in. x in.)	(3) Equation (23)	(4) K	(5) Failure mode Applicable	(6) $\frac{T_{cal.}}{T_{expt.}}$	Error in Calculated Torque (7) %age error	Error in Calculated Torque (8) Mean %age error
HB/2	$6\frac{1}{16} \times 7\frac{5}{8}$	1.10	0.50	I	0.51	49	
HB/3	$6 \times 7\frac{1}{2}$	1.65	0.27	I	0.82	18	
HB/4	$6 \times 7\frac{3}{4}$	2.32	0.19	I	0.90	10	
HB/5	$6 \times 7\frac{1}{2}$	2.60	0.16	I	0.84	16	
HB/8	6×9	1.62	0.27	I	0.94	6	12
HB/9	6×9	1.95	0.21	I	0.91	9	
HB/10	$6 \times 9\frac{1}{16}$	2.18	0.19	I	0.91	9	
HB/11	6×9	2.64	0.15	I	0.90	10	
HB/13	6×12	0.94	0.50	I	0.95	5	
HB/15	6×12	1.47	0.27	I	0.97	3	
HB/16	6×12	2.04	0.18	I	0.84	16	
HB/17	6×12	2.50	0.14	I	0.81	19	

SOLID R.C.C. RECTANGULAR BEAMS SUBJECTED TO COMBINED BENDING TORSION AND SHEAR

SOURCE : LYALIN(33)

TABLE - 9 Taken from Reference No. 33

(1) Beam No.	(2) b x h (cm. x cm.)	(3) $K = \frac{T}{M}$	(4) $\lambda = \frac{2T}{Qb}$	(5) Failure Mode Applicable	(6) $\frac{(\frac{M}{T})_{Expt.}}{(\frac{M}{T})_{Theoretical}}$	(7) %age error	(8) Mean %age error.
1.	20 x 31	0.10	1.05	I	1.03	3	
2.	20 x 31	0.10	1.05	I	1.09	9	
3.	20 x 30	0.20	2.10	I	0.99	1	
4.	20 x 31	0.20	2.10	I	1.01	1	
5.	20 x 31	0.38	4.00	I	0.88	12	
6.	20 x 30½	0.40	4.20	I	0.88	12	
7.	20 x 30½	0.20	2.10	I	1.06	6	
8.	20 x 30½	0.20	2.10	I	1.00	0	
9.	20 x 31	0.20	2.10	I	1.07	7	
10.	20 x 31	0.20	2.10	I	1.06	6	
11.	22 x 20	0.20	1.91	I	1.10	10	
12.	22 x 20	0.20	1.64	I	1.03	3	

TABLE - 9 (Contd.)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
13	22 x 20	0.20	1.95	I	1.05	5	
14	22 x 30	0.52	1.91	I	1.12	12	
15	22 x 30	0.40	2.54	I	0.93	7	
16	22 x 30	0.40	2.54	I	0.93	7	
17	22 x 30	0.47	2.98	I	1.15	15	
18	19 x 30	1.00	11.00	II	0.80	20	8.7
19	20 x 30	1.00	11.00	II	0.79	21	
20	15 x 33	0.19	1.32	I	0.85	15	
21	15 x 23	0.20	1.40	I	0.89	11	

calculated by equations Nos. (23) or (24) whichever corresponds to the failure mode shown in the fifth column. The sixth column in Table Nos. (1 - 7) gives the ratio $\frac{T_{calculated}}{T_{experimental}}$. The seventh column gives the percentage error in theoretical torque capacity, in relation to the corresponding actual torque capacity. The eight column givesthe mean persentage error involved in theoretical torque capacity, for the whole series of the specimens tested by one researcher. Table Nos. (1 - 9) have been prepared by subdividing the test results of each researcher in the following sub-headings :

(i) Pure torsion (Table Nos. 1 - 4)

(ii) Torsion combined with bending (Table Nos. 5 - 8)

(iii) Torsion combined with bending and shear (Table No. 9)

6.3.3.2 INFERENCES DRAWN FROM THE TABLES :

(a) Pure torsion (Solid and Hollow R.C.C. rectangular Sections) :

(i) The second mode of failure is found to give the minimum value of failing torque capacity, and so invariably applicable in case of pure torsion.

(ii) The theoretical torque capacity is generally on the unconservativ side, and this average error is about 17%.

(iii) The actual ratio between longitudinal and web steel provided in the tests, generally satisfies equation (24), although the upper limit is not always satisfied.

(iv) As the Lessig's theory which is the basis for Russian Code overestimates the torsional capacity of specimens, under pure torsion, a capacity reduction factor of 0.8 is suggested.

Also, Rausch and Cowan's theories which are the basis of German and Australian Codes respectively have shown to be nonconservative by Hsu (Fig. 4.2). Hsu[55] recommends an expression for under reinforced beams

$$T = T_o + \Omega \frac{b' h' F_{s1} \sigma_s}{u}$$

where T_o = ultimate twisting capacity of an equivalent plain concrete specimen in pure torsion

Ω = a constant

= 1.20, based on experiments conducted by Hsu[55]

b',h' = breadth and depth respectively of the concrete core within web stirrups.

(b) Combined bending and torsion for R.C.C. rectangular (both solid and hollow) sections : (Table Nos. 5 - 8).

(i) For values of $K \leq 1$, failure mode 1 is found to be generally applicable. The error is found to be about 8% on the non-conservative side, although for K = 1 there were tests in which the Lessig's theory was valid.

(ii) The actual ratio between web and longitudinal steel provided in the test specimens, generally satisfies equation (23) for $K \leq 1$.

(iii) The theoretically calculated failing torque capacities are found to be, generally, on the conservative side for values of $K \leq 1$ for hollow sections (Table No. 7).

(iv) From Table 8 given by Collins et. al.[89] Yudin's theory is found to give results on the conservative side.

(c) Torsion combined with bending and shear : (Table No. 9)

(i) The theoretical values of failing torque, according to mode indicated in the fifth column of Table No. 9, are found to be very close to the corresponding test-values, in case $K \leq 0.2$ and $\lambda \leq 4$ and the error is on the safe side. For K greater than above and less than 1.0, and $\lambda > 4$ the calculated values are on the unsafe side with an error of about 9% . All these beams are reported to have failed in mode 1.

(ii) The second failure mode is found to be applicable for theoretical analysis when $K > 0.2$ and $\lambda > 4$ and the theory gives values on the unsafe side by about 20%.

6.3.3.3 SIMPLIFIED LESSIG'S ULTIMATE EQUILIBRIUM METHOD :

Based on the observations and calculations of Table Nos. (1 - 9) it is found that the depths of the compression stress block x_1 and x_2, in failure modes 1 and 2 respectively, calculated by equations (15) and (19), can be approximated to $0.1h_o$ and $0.1b$ without disturbing Lessig's analysis.

With the above simplifications, Lessig's equations reduce to the following simplified forms :

Failure Mode 1 :

$$T\left[\frac{C_1}{b} + \frac{1}{K}\right] \leq \left[\sigma_s F_{s1} + \sigma_{st} \cdot \frac{f_{st} \cdot C_1^2}{u(2h + b)}\right] .0.95\, h_o \quad \ldots\ldots \quad (27)$$

$$x_1 = 0.1\, h_o \quad \ldots\ldots\ldots \quad (28)$$

$$C_1 = -\frac{b}{K} + \left[\left(\frac{b}{K}\right)^2 + \frac{\sigma_s F_{s1} u\,(2h + b)}{\sigma_{st} f_{st}}\right]^{\frac{1}{2}} \quad \ldots\ldots \quad (29)$$

$$\text{and} \quad C_1 \leq 2h + b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

Failure Mode 2 :

$$T.\frac{C_2}{h}\left(1+\frac{1}{\lambda}\right) \leq \left[\sigma_s F_{s2} + \sigma_{st}\cdot \frac{f_{st} C_2^2}{u(2b+h)}\right](0.95b - a) \qquad (31)$$

$$x_2 = 0.1b \quad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (32)$$

$$C_2 = \left[\frac{\sigma_s F_{s2}\ u}{\sigma_{st} f_{st}}(2b+h)\right]^{\frac{1}{2}} \quad \ldots\ldots\ldots \qquad (33)$$

$$\text{and } C_2 \leq 2b + h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (34)$$

The predicted failure torque is then taken as the lesser of the two values, obtained one each, from equations (27) or (31).

6.4 DESIGN METHOD FOR R.C.C. SPECIMENS SUBJECTED TO COMBINED LOADING, BASED ON, LESSIG'S SIMPLIFIED ULTIMATE EQUILIBRIUM ANALYSIS :

In the ultimate load design methods, there are different philosophies in incorporating :

(1) Factors of safety or overload factors or safety factors,

(2) Strength and variation in strength due to the variation in material properties in the field,

(3) Serviceability

(4) Economy

(5) Ductility and warning before actual failures.

In the U.S. specifications (American Concrete Institute Building Code ACI 318 - 63) a capacity reduction factor Φ is used for variation in strength of members and their mode of failure, and overload factors for design loads the value of which is 1.5 for dead load and 1.8 for live loads when effects of wind and earthquakes can be neglected. The value of Φ is taken as 0.9 for R.C. beams in pure bending so

as to fail in secondary compression failure mode, 0.85 for shear and bond (the shear strength is conservatively taken as the force at the first diagonal cracking of concrete with yielding of stirrups), 0.70 for tied compression members since their failure is sudden and the strength depends upon the strength of concrete.

In Indian Standard Specifications (IS456 : 1964) the stress in stress block for design purposes is assumed to be 80% of the theoretical ultimate stress in bending (R_b) to take care of the variation of concrete strength in the field. The value of $R_b = 0.8 \times .85 \times .8\ \sigma_{cu} = 0.55\ \sigma_{cu}$ where σ_{cu} = cube strength of concrete. The overload factors are 1.5 for dead load and 2.2 for live loads for the case when wind and earthquake effects are neglected.

A third approach of ultimate load design is called "Limit State Design" used in Russia and recommend by International Committee for concrete (CEB, Paris). Here the characteristic strengths are divided by safety factor γ_m to give design strengths (the value of γ_m for concrete is about 1.5 - 1.8 and for steel is about 1.1 to 1.25 because of better quality control in steel) and the characteristic or normative loads are increased by γ_ℓ to give design loadings (γ_ℓ is less for dead loads, about 1.1, more for live loads about 1.3 - 1.4 , and this coefficient includes risk of failure, quality of construction and the reliability of assumptions made in the calculations). The total factor of safety, $\gamma = \gamma_m \gamma_\ell$.

In addition to the checks for strength, adequate checks for cracking and deflections at working load are also made in this method.

These three approaches to ultimate load design are interchangeable. However, the third approach based on semi-probabilistic considerations and divisions of safety factors between various considerations, seems to be the most logical.

The design method given in the subsequent sub-section is based on the Russian approach of Calculated Limit States Method[51]. For converting the ultimate compressive strength of concrete, to its corresponding strength at design level, the following relation is used :

$$(R_b)_{design} = (R_b)_{ultimate} \times 0.55$$

where 0.55 is the load factor based on experiments. For converting the yield strengths of reinforcements, to their corresponding design values, the following relation is used :

$$(\sigma_{st}, \sigma_s)_{design} = 0.8\ (\sigma_{st}, \sigma_s)_{yield},$$

where 0.8 is the conversion factor. For converting the ultimate external loading, to its corresponding design value, the following relation is used :

$$(M,\ T,\ or\ Q)_{ultimate} = \quad (M, T,\ or\ Q)_{design},$$

where γ_ℓ may be taken as 1.4 on an average. If the dead and live load are known, the ultimate design loads are found by multiplying with the corresponding values of γ_ℓ.

Here-after, all values of strengths of concrete and steel and the externally applied loading, are meant to be the design values.

6.4.1 EXTRACTS FROM ULTIMATE LOAD THEORY FOR R.C.C. RECTANGULAR BEAMS SUBJECTED TO PURE FLEXURE :

For a singly reinforced rectangular R.C. beam in pure flexure, reinforced with mild steel bars such that concrete crushes in the compression zone after the yielding of the tensile steel, gives sufficient warning before failure called "secondary compression failure", the following expressions are valid at ultimate :

(1) From equilibrium using a rectangular stress block with a stress of R_b in the beam, the depth of the stress block x is given by :

$$x = \frac{F_{s1}\ \sigma_s}{R_b\ b} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (35a)$$

where F_{s1} = Amount of tensile steel

σ_s = Yield stress of steel

b = Width of the cross-section.

If h_o is the effective depth, a dimensionless parameter α can be defined as : $\alpha = \frac{x}{h_o}$. The depth of the neutral axis can be taken as $x_{na} = \frac{x}{K_1}$ where K_1 is taken 0.85 for $f_c' \leq 281$ kgCm^2. (ACI 318-63)

(2) Taking moments about the tensile steel, the following is obtained :

$$M_{ult.} = R_b \cdot b\ x\ (h_o - \frac{x}{2})$$

$$= R_b\ b\ h_o^2 \alpha\ (1 - 0.5\ \alpha) \qquad \ldots\ldots \qquad (35b)$$

In a simpler form, it may be written as :

$$M_{ult.} = F_{s1}\ \sigma_s\ (h_o - \frac{x}{2})$$

Eqn. (35b) is preferred by practising engineers because it is of the form used in working stress method and may be written in a form given by Baikov[51] as :

$$M = A_o\ b\ h^2_o\ R_b \quad \ldots\ldots\ldots\ldots \quad (35c)$$

where $$A_o = \alpha (1 - 0.5\ \alpha\ ) \quad \ldots\ldots\ldots \quad (36)$$

Also $\alpha < \alpha_{max}$ (This condition is required to be satisfied in order that the failure may be a secondary compression failure).

The value of α_{max} for M150 and M200 concrete is 0.55 . For other types of concrete, Baikov[51] may be referred. The optimum value of α , to be assumed in the initial stages of design, is $\alpha = 0.3$ to 0.4 , based on experiments conducted in Russia[51].

The equation (35c) is valid only if the tensile steel yields. To satisfy this criterion, the amount of steel should be less than F_{s1b} (balanced steel at ultimate) where

$$F_{s1b} = b\ h_o \frac{R_b}{\sigma_s} \left(K_1 \cdot \frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u} \right) \text{ where :}$$

ε_u = Ultimate crushing strain in concrete,

ε_y = Yield strain in steel

K_1 = Ratio between area of the stress block to an equivalent rectangular stress block.

The quantity within the bracket is usually around 0.4 .

In order to increase this amount, for the same section, compression steel F_{sc} of yield stress σ_s is added such that :

$$F_{s1} - F_{sc} \leq F_{s1b} \cong 0.4\ b\ h_o \frac{R_b}{\sigma_s}$$

$$\text{Also} \quad F_{s1} = \frac{M}{\gamma_o h_o \sigma_s} \quad \ldots\ldots\ldots\ldots \qquad (37)$$

where $\quad \gamma_o = 1 - 0.5\alpha \quad \ldots\ldots\ldots\ldots \qquad (38)$

and $\quad h_o = \Upsilon_o \left(\frac{M}{b R_b} \right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots \qquad (39)$

where $\quad \Upsilon_o = \frac{1}{(A_o)^{1/2}} \quad \ldots\ldots\ldots\ldots \qquad (40)$

Table No. (10) which has been taken from Baikov[51], gives the values of the coefficients A_o, Υ_o and γ_o, corresponding to various values of α.

6.4.2 DESIGN STEPS FOR A RECTANGULAR (SOLID OR HOLLOW), T OR L SECTION, R.C.C. BEAMS SUBJECTED TO COMBINED BENDING, TORSION AND SHEAR, OR COMBINED BENDING AND TORSION OR PURE TORSION:

STEPS :

1. For fixing up the section of the beam, i.e. breadth b and overall depth h, b is generally fixed from the constructional and architectural point of view, say, b may be the width of the column on which the beam rests. To fix up h, it is taken as the maximum of the following :

(1) To prevent excessive crack formation :

$T \leq 0.07\, R_b\, b^2\, h$ . . . . same as eqn. (26)

here T = External twisting moment at design level

R_b = design compressive strength of concrete

0.07 = empirical factor obtained from experiments conducted in Russia[46].

TABLE 10

α	Υ_0	γ_0	A_0	α	Υ_0	γ_0	A_0
1	2	3	4	5	6	7	8
0.01	10.00	0.995	0.010	0.29	2.01	0.855	0.248
0.02	7.12	0.990	0.020	0.30	1.98	0.850	0.255
0.03	5.82	0.985	0.030	0.31	1.95	0.845	0.262
0.04	5.05	0.980	0.039	0.32	1.93	0.840	0.269
0.05	4.52	0.975	0.048	0.33	1.90	0.835	0.275
0.06	4.15	0.970	0.058	0.34	1.88	0.830	0.282
0.07	3.85	0.965	0.065	0.35	1.86	0.825	0.289
0.08	3.61	0.960	0.077	0.36	1.84	0.820	0.295
0.09	3.41	0.955	0.085	0.37	1.83	0.815	0.301
0.10	3.24	0.950	0.095	0.38	1.80	0.810	0.309
0.11	3.11	0.945	0.104	0.39	1.78	0.805	0.314
0.12	2.98	0.940	0.113	0.40	1.77	0.800	0.320
0.13	2.88	0.935	0.121	0.41	1.75	0.795	0.326
0.14	2.77	0.930	0.130	0.42	1.74	0.790	0.332
0.15	2.68	0.925	0.139	0.43	1.72	0.785	0.337
0.16	2.61	0.920	0.147	0.44	1.71	0.780	0.343
0.17	2.53	0.915	0.155	0.45	1.69	0.775	0.349
0.18	2.47	0.910	0.164	0.46	1.68	0.770	0.354
0.19	2.41	0.905	0.172	0.47	1.67	0.765	0.359
0.20	2.36	0.900	0.180	0.48	1.66	0.760	0.365
0.21	2.21	0.895	0.188	0.49	1.64	0.755	0.370
0.22	2.26	0.890	0.196	0.50	1.63	0.750	0.375
0.23	2.22	0.885	0.203	0.51	1.62	0.745	0.380

1	2	3	4	5	6	7	8
0.24	2.18	0.880	0.211	0.52	1.61	0.740	0.385
0.25	2.14	0.875	0.219	0.53	1.60	0.735	0.390
0.26	2.10	0.870	0.226	0.54	1.59	0.730	0.394
0.27	2.07	0.865	0.234	0.55	1.58	0.725	0.400

(ii) From Pure Flexural considerations :

$$h_o = \gamma_o \left(\frac{M}{b R_b} \right)^{\frac{1}{2}} \quad \ldots\ldots\ldots \quad \text{same as eqn. (39)}$$

for finding γ_o, assume $\alpha_{optimum} = 0.30$ to 0.35

(iii) For $K \geqslant 0.2$,

$$h \leq 2b \qquad \text{(From Reference. 33)}$$

2. F_{s1} is calculated from :

$$F_{s1} = \frac{M}{\gamma_o h_o \sigma_s} \quad \ldots\ldots \quad \text{same as eqn.(37)}$$

3. (i) First it is checked from the following relation, whether, any transverse steel is required or not : If the relation is satisfied, then no transverse steel is required :

$$T \angle 0.15 R_t b^2 (3h - b) \quad \ldots\ldots \quad \text{same as eqn. (25)}$$

where R_t = design tensile resistance of concrete.

(ii) to fix up spacing u of the stirrups, the following condition must be observed :

$$u \leq b - 2a \quad \ldots\ldots\ldots \quad \text{From Reference. 46}$$

This condition is based on the consideration that spacing between the stirrups must be such as to ensure that at least one branch of a stirrups crosses each crack running at 45° to the axis of the beam; a, the uniform concrete cover, is assumed.

(iii) To find the area of cross-section f_{st} of the stirrup the following relation is to be used :

$$\frac{\sigma_{st} \cdot f_{st} \, b}{\sigma_s \quad F_{s1} \quad u} \left[1 + \frac{2}{K} \left(\frac{b}{2h+b} \right)^{\frac{1}{2}} \right] = 1 \text{ to } 1.5$$

..... same as eqn.(23)

Nominal longitudinal compressional steel is provided at the top horizontal face of the section for binding the stirrups.

4. Checking the strength of the designed section, against Failure mode 1 : (i) First C_1 is calculated from :

$$C_1 = -\frac{b}{K} + \left[\left(\frac{b}{K}\right)^2 + \frac{\sigma_s \; F_{s1} \; u}{\sigma_{st} \; f_{st}} \cdot (2h + b) \right]^{\frac{1}{2}}$$

...... same as eqn.(16)

The calculated C_1 must satisfy :

$$C_1 \leq 2h + b \quad \ldots\ldots\ldots$$

same as eqn.(17)

If C_1 does not satisfy the above inequality, then checking for failure mode 1 is not required.

(ii) The depth of compression zone x_1 is calculated from

$$x_1 = 0.1 \, h_o \quad \ldots\ldots\ldots$$

same as eqn. (28)

x_1 , thus calculated must satisfy :

$$x_1 \leq (0.55 - 0.7 \sqrt{K}) \, h_o \quad \ldots\ldots\ldots$$

same as eqn. (22)

(iii) Then the strength of the designed section is adequate against failure mode 1, if the following condition is satisfied :

$$T\left(\frac{C_1}{b} + \frac{1}{K}\right) \leq \left[\sigma_s F_{s1} + \sigma_{st} \cdot \frac{f_{st} \cdot C_1^2}{u(2h + b)}\right] X\ 0.95\ h_o$$

..... same as eqn. (27)

5. Checking the design strength against Failure Mode 2 :

If $\lambda \angle 1 - 2\frac{a}{b}$, the designed strength of section need not be checked against the requirements of Failure Mode 2. This inequality is based on experiments conducted by Lessig[30,31]. If $\lambda \geqslant 1 - \frac{2a}{b}$, checking of the designed strength according to Failure Mode 2 is required as given below :

(i) First C_2 is calculated from :

$$C_2 = \left[\frac{\sigma_s F_{s2}\ u}{\sigma_{st}\ f_{st}} \cdot (2b + h)\right]^{\frac{1}{2}} \quad \text{..... same as eqn.(33)}$$

C_2, thus calculated, must satisfy the following condition :

$$C_2 \leq 2b + h \quad \text{.................... same as eqn. (34)}$$

If C_2 does not satisfy the above inequality, checking of strength of the section according to Failure Mode 2 is not required. If C_2 satisfies the above condition, then checking of strength against Failure Mode 2 is carried out as follows :

(ii) The depth, x_2, of the compression zone is calculated from the following relation :

$$x_2 = 0.1b \quad \text{.......... same as eqn. (32)}$$

(iii) The strength of the section is sufficient, against failure mode 2, if the following relation is satisfied :

$$T \cdot \frac{C_2}{h}\left(1 + \frac{1}{\lambda}\right) \leq \left[\sigma_s F_{s2} + \sigma_{st} \cdot \frac{f_{st} \cdot C_2^2}{u(2b + h)} \right] (0.95\ b - a)$$

..... same as eqn.(31)

(iv) Finally, check against the distribution of longitudinal and web steel, as required in the Failure mode 2, is carried out as follows :

$$0.5 \leq \frac{\sigma_{st} \cdot f_{st} \cdot h}{\sigma_s F_{s2}\ u} \leq 1.5 \qquad \text{.......} \qquad \text{same as eqn. (24)}$$

6. Reinforced concrete beams subjected to combined loading must always be checked for pure bending regardless of torsion. The same thing holds for shear.

7. T AND L SECTIONS:

Upto-date there hasbeen no experimentation reported for T and L sections of R.C.C. beams subjected to combined loading. Therefore, in the absence of any experimental data, the design of T and L sections subjected to combined stresses can be done exactly like that for an equivalent rectangular section by neglecting the over hanging flanges.

8. For pure torsion in R.C. sections, a capacity reduction factor of 0.8 is suggested, based on the observations of Table Nos.(1 - 9).

Now, a design problem on R.C.C. rectangular section subjected to combined loading, shall be solved in the next sub-section, so as to illustrate the use of design steps cited above.

6.5 ILLUSTRATIVE DESIGN EXAMPLE :

In Ref. 46 an example of analysis is given. Herein an example is given in which the cross section is first fixed with respect to pure flexural considerations.

PROBLEM :

Design a rectangular R.C.C.beam sectionsubjected to a design bending moment of 3820 kg. m., a design twisting moment of 840 kg.m. and a design transverse shear force of 2100 Kg. The design resistances of concrete and steel are as :

$$R_t = 7 \text{ kg./cm}^2$$
$$R_b = 100 \text{ kg./cm}^2$$
$$\sigma_s = 3600 \text{ kg/cm}^2$$
$$\sigma_{st} = 2850 \text{ kg./cm}^2$$

The width b of the section = 20 cm., is fixed from architectural considerations.

SOLUTION :

(1) h is taken as maximum of the following :

(i) To prevent excessive crack formation :

$$T \leq 0.07 \, R_b \, b^2 \, h$$

or $$h \geqslant \frac{T}{0.07 \, R_b \, b^2}$$

$$\geqslant \frac{84000}{0.07 \times 100 \times 400}$$

$$\geqslant 30 \text{ cm.}$$

(ii) From pure flexural consideration :

$$h_o = r_o \left(\frac{M}{bR_b} \right)^{\frac{1}{2}}$$

for finding r_o, assume $\alpha_{optimum} = 0.3$

From Table No.(1) , against the value of $\alpha = 0.3$, the corresponding value of $r_o = 1.98$

$\therefore h_o = 1.98 \left(\frac{382000}{20 \times 100} \right)^{\frac{1}{2}}$

$= 27.5$ cms.

(iii) In this problem $K = \frac{T}{M}$

$= \frac{840}{3820}$

$= 0.22$

so for $K \geq 0.2$,

$h \leq 2b$

So $h \leq 2 \times 20$

≤ 40 cms.

So the minimum value of h = 31 cms, with a uniform concrete cover 'a' = 3.5 cms.

2. $F_{s1} = \frac{M}{\gamma_o \, h_o \, \sigma_s}$

Now $A_o = \frac{M}{b \, h_o^2 \, R_n}$

$= \frac{382000}{20 \times (27.5)^2 \times 100}$

$= 0.253$

$A_o = 0.253$ corresponds to $\gamma_o = 0.850$

and $\alpha = 0.30$ From Table No. 10

So $\alpha < \alpha_{max}$

or $0.3 < 0.55$ ....(O.K.)

$$\therefore \; F_{s1} = \frac{382000}{0.85 \times 27.5 \times 3600}$$

$$= 4.55 \text{ sq. cm.}$$

Provide 2 bars of 18 mm. diameter each as longitudinal tension steel.

Actual cross-sectional area of longitudinal steel provided

$$= 5.10 \text{ sq. cms.}$$

Provide 2 bars of 12 mm. diameter each, one in each corner of the top face of width b, as nominal reinforcement, for binding the stirrups.

3. To check the requirement of transverse steel, if the following relation is satisfied, no transverse steel is required, otherwise, it is required :

$$T < 0.15 \; R_t \; b^2 \; (3h - b)$$

$$\text{or} \quad T < 0.15 \times 7 \times 400 \times (93 - 20) = 30700$$

$$\text{or} \quad 84000 \not> 30700$$

So the relation is not satisfied, and therefore stirrups are required.

(ii) to fix up the spacing of the stirrups, the following condition must be satisfied :

$$u \leq b - 2a$$

$$\leq 20 - 7$$

$$\leq 13 \text{ cms.}$$

Provide the spacing, u, as = 8 cms. c/c

(iii) Area of cross-section, f_{st}, is found from :

$$\frac{\sigma_{st} \cdot f_{st} \; b}{\sigma_s (F_{s1}) \cdot u} \cdot \left[1 + \frac{2}{K} \left(\frac{b}{2h+b} \right)^{\frac{1}{2}} \right] = 1.08$$

$$\text{or} \quad \frac{2850 \times f_{st} \times 20}{3600 \times 5.1 \times 8} \left[1 + \frac{2}{0.22} \left(\frac{20}{82} \right)^{\frac{1}{2}} \right] = 1.08$$

$$\text{or} \quad \frac{2800 \times f_{st} \times 20}{3600 \times 5.1 \times 8} \left[1 + 9.1 \times 0.49 \right] = 1.08$$

$$\text{or} \quad f_{st} = \frac{1.08 \times 3600 \times 5.1 \times 8}{2850 \times 20 \times 5.47}$$

$$= 0.51 \text{ sq. cm.}$$

Provide 8 mm. diameter stirrups @ 8 cm. c/c.

Actual cross-sectional area of stirrups material provided is = 0.51 sq.cm.

The section is fully designed, now, and it is to be checked for strength according to Failure modes 1 and 2.

4. (i) C_1 is calculated from

$$C_1 = -\frac{b}{K} + \left[\left(\frac{b}{K}\right)^2 + \frac{\sigma_s F_{s1} \; u}{\sigma_{st} \; f_{st}} \cdot (2h + b) \right]^{\frac{1}{2}}$$

$$= -\frac{20}{0.22} + \left[\left(\frac{20}{0.22} \right)^2 + \frac{3600 \times 5.1 \times 8}{0.51 \times 2850} \times 82 \right]^{\frac{1}{2}}$$

$$= -90.8 + \left[8300 + 8300 \right]^{\frac{1}{2}}$$

$$= -90.8 + 129$$

$$= 38.2 \text{ cm.}$$

Also $\quad C_1 \leq 2h + b$

or $38.2 \leq 82$ which is satisfied.

(ii) The depth of the compression zone x_1 is calculated from :

$$x_1 = 0.1\ h_o$$

$$= 0.1 \times 27.5$$

$$= 2.75 \text{ cm.}$$

(iii) Finally checking the strength of the designed section, according to Failure Mode 1, by the following relation :

$$T\left[\frac{C_1}{b} + \frac{1}{K}\right] < \left[\sigma_s F_{s1} + \sigma_{st} \cdot \frac{f_{st} \cdot C^2{}_1}{u(2h + b)}\right] \times 0.95\ h_o$$

First calculate the value of left hand side of this expression,

$$T\left[\frac{C_1}{b} + \frac{1}{K}\right] = 840\left[\frac{38.2}{20} + 4.54\right]$$

$$= 840 \times 6.45 \times 100 \text{ kg. cm.}$$

$$= 5410{,}00 \text{ kg. cm.}$$

Now, the value of right-hand side of the above expression is calculated :

$$\left[\sigma_s F_{s1} + \sigma_{st} \cdot \frac{f_{st}\ C^2{}_1}{u\ (2h + b)}\right] \times 0.95\ h_o =$$

$$= \left[3600 \times 5.1 + 2850 \times \frac{0.51 \times (38.2)^2}{8 \times 82}\right] \times 0.95 \times 27.5$$

$$= \left[18400 + 3240\right] \times 26.1$$

$$= 21640 \times 26.1$$

$$= 565000 \text{ kg. cm.}$$

so $\quad 541000 < 565000$

So the strength of the designed section is sufficient according to Failure Mode 1.

5. Checking the strength of the designed section, according to Failure Mode 2 :

(i) $\lambda = \frac{2T}{Qb}$

$= \frac{2 \times 840}{2100 \times 20}$

$= 4$

so $\lambda > 1 - 2\frac{a}{b}$

or $4 > 1 - \frac{2 \times 3.5}{20}$

or $4 > 0.65$

So, Failure Mode 2 is possible if C_2 also satisfies the requirements of this mode.

(ii) C_2 is calculated from :

$$C_2 = \left[\left(\frac{\sigma_s F_{s2} u}{\sigma_{st} f_{st}} \right) . (2b + h) \right]^{\frac{1}{2}}$$

$$= \left[\frac{3600 \times 3.68 \times 8}{2850 \times 0.51} \times 71 \right]^{\frac{1}{2}}$$

$$= (5200)^{\frac{1}{2}}$$

Also C_2 has to satisfy :

$$C_2 \leq 2b + h$$

or $72 \leq 71$ which is not satisfied so checking of strength according to Failure Mode 2 is not required.

Hence the designed section is alright as regards the restrictions due to combined loading. Also, as there is a shear force of 2100 kg., it can be seen on checking that no transverse reinforcement, other than already provided, is required for the exclusive effect of shear force.

The designed section is shown in (Fig. 6.5).

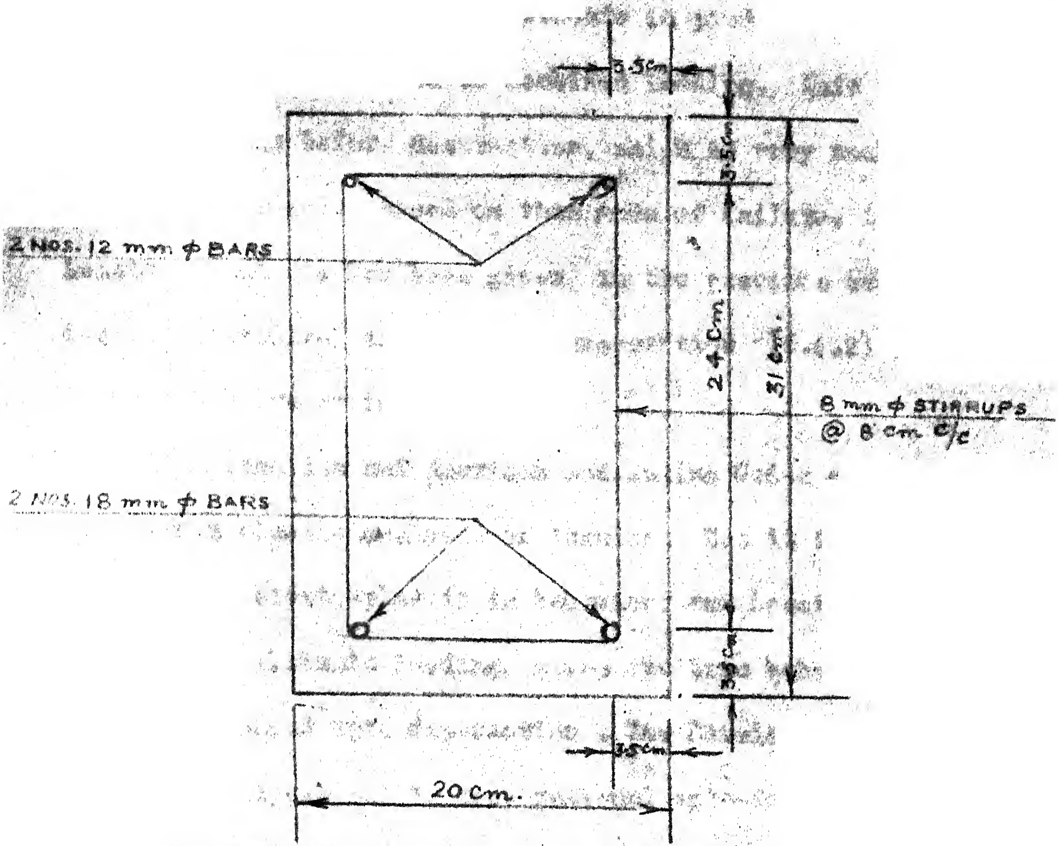


FIG. 6.5 Designed Section

SECTION 7 : CONCLUDING REMARKS

7.1 RECOMMENDATIONS FOR INDIAN STANDARD CODE OF PRACTICE :

Lessig's Analysis according to failure mode 1 is most authentic, logical and rational method uptodate in predicting the strength of R.C.C. members subjected to combined loading. This failure mode gives enough warning before destruction, which is very much desirable in any design procedure . Based on this mode of failure, the Simplified Lessig's Analysis has been given in the previous section. The proposed design procedure given in the sub-section (6.4.2) are recommended to be incorporated in the code.

Australian and American and Indian Codes of practice are based on Cowan's elastic analysis of torsion. But it is established that concrete is elasto-plastic in behaviour and Lessig's Analysis which is based on ultimate loading, gives the true behaviour of R.C.C. specimens loaded upto destruction . The Russian Code is based on Lessig's analysis and is most rational upto-date. The Simplified Design Procedure proposed herein, is based on Russian Code of practice.

7.2 RECOMMENDATIONS FOR FURTHER STUDY :

The following areas of study are yet to be further elaborated with more experimentations :

(i) Study of failure modes under combined loading for R.C.C. specimens (Rectangular, T and L Sections),

(ii) Cases in which failure takes place before reinforcement yields i.e. primary compression failure cases,

(iii) Continuous T, L, rectangular beams, grid floors, slabs, under combined loading,

(iv) Shear failures (modes 2 and 3) which are sudden and destructive ,

(v) R.C.C. specimens under combined loading in which $K \geq 1$,

(vi) Fixation of ratio between areas of longitudinal and web steels in R.C.C. specimens subjected to combined loading, so that the failure may be a secondary compression failure,

(vii) Yudin's generalisation of Lessig's Analysis,

(viii) Buchanan's Non-Linear Theory of Elasticity for plain concrete specimens, under pure torsion, to be extended to R.C.C. specimens under combined loading.

7.3 CONCLUSIONS :

(i) The use of the plastic theory to predict the strength of plain prismatic specimens under pure torsion yields a safe and satisfactory correlation with the experimental evidence,

(ii) The plain concrete specimens under pure torsion fail at the appearance of first crack,

(iii) Buchanan's non-linear theory of elasticity for plain concrete specimens subjected to pure torsion is the most elaborate analysis uptodate and very much promising. The torsional strength of a beam with longitudinal steel only should be assumed to be the same as a beam without reinforcement,

(iv) The strength of beams reinforced longitudinally only and subjected to combined bending and torsion depends on; the strength of concrete in compression and tension combined

with other stresses, crack propogation, dowel forces as a function of concrete or steel shear stresses and perhaps aggregate interlock. The problem is somewhat similar to combined shear and flexure in its difficulties. There appears to be little interaction between bending and torsional moment for such specimens until either or both the bending moment and torque exceed 50 per cent of the strength of the specimen in pure bending and pure torsion respectively. Lessig's Ultimate Equilibrium Method is invalid for such beams,

(v) For beams reinforced with both longitudinal and web steel, and subjected to pure torsion, tensile cracks appear on the face of the beams with an inclination to the twist-axis of the beam of approximately 45°, when the twisting moment is approximately the same as the cracking torque of a similar beam of plain concrete. Once the member has cracked, the behaviour beyond this stage depends primarily on the amount and position of the reinforcement. Such beams most likely fail by mode 2. Lessig's Ultimate Equilibrium Method over-estimates the capacity of such beams and a capacity reduction factor of 0.8 is suggested while ucing Lessig's Analysis, in the absence of elaborate experimental data.

Hsu's Analysis for such beams gives more correct strength predictions. Lessig's Analysis for mode 2 is yet to be modified and brought within conservative limits.

(vi) For R.C.C. rectangular beams subjected to combined bending and torsion, Lessig's Analysis according to failure mode 1 is invariably applicable when $K < 1$ and transverse shear force is negligible. For cases where $K \geqslant 1$, further experimentation is needed to arrive at a conservative and rational analysis and also, the failure mode is yet to be established for such cases. In the absence of elaborate experimentation, T and L, R.C.C. sections can be treated as equivalent to rectangular sections by ignoring the overhanging flanges.

(vii) For R.C.C. beams (Rect., T, L) subjected to combined loading, the Ultimate Equilibrium Method has been found to have given conservative results when K   1 and when shear force is not substantial. In this loading combination, first mode of failure invariable occurs.

(viii) As the failure mode 1 is a secondary compression failure and it gives sufficient warning before destruction, the design of R.C.C. sections subjected to combined loading is recommended to be carried out according to this failure mode.

(ix) The Simplified Lessig's Analysis, suggested by the author is fairly convenient in using it in design offices.

(x) Preliminary study of Yudin's generalisation of Lessig's Analysis appears to be more close and conservative when compared with the few test values available in the literature so far.

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